<u>$n=4,\ p=5$</u>.

$$w_0 \cdot -2\rho = (0,0,0)$$
$$w_0 s_0 \cdot -2\rho = (2,0,2)$$
$$w_0 s_0 s_1 \cdot -2\rho = (3,1,1)$$
$$w_0 s_0 s_3 \cdot -2\rho = (1,1,3)$$
$$w_0 s_0 s_1 s_3 \cdot -2\rho = (2,2,2)$$
$$w_0 s_0 s_1 s_2 \cdot -2\rho = (5,0,1)$$
$$w_0 s_0 s_3 s_2 \cdot -2\rho = (1,0,5)$$
$$w_0 s_0 s_1 s_3 s_2 \cdot -2\rho = (3,2,3)$$

give all weights (of the form $w_0 w \cdot -2\rho$ with $\ell(w_0) = \ell(w_0 w) - \ell(w)$) which are less than or equal to some restricted weight.
(In fact it's true for any $p \geq 5$).

We use the recurrence formula

$$C_{w_0 us} = C_{w_0 u} C_s - \sum_{w_0 vs \leq w_0 v < w_0 u} \mu(w_0 v, w_0 u) C_{w_0 v}$$

to compute the corresponding KL-polynomials: first compute

$$C_{w_0} = t^{-6} \hat{T}.$$
$$C_{w_0 s_0} = t^{-7}(\hat{T} + \hat{T}_{s_0})$$
$$C_{w_0 s_0 s_1} = t^{-8}(\hat{T} + \hat{T}_{s_0} + \hat{T}_{s_0 s_1})$$
$$C_{w_0 s_0 s_3} = t^{-8}(\hat{T} + \hat{T}_{s_0} + \hat{T}_{s_0 s_3})$$
$$C_{w_0 s_0 s_1 s_3} = t^{-9}\left((1+t^2)\hat{T} + \hat{T}_{s_0} + \hat{T}_{s_0 s_1} + \hat{T}_{s_0 s_3} + \hat{T}_{s_0 s_1 s_3}\right)$$
$$C_{w_0 s_0 s_1 s_2} = t^{-9}(\hat{T} + \hat{T}_{s_0} + \hat{T}_{s_0 s_1} + \hat{T}_{s_0 s_1 s_2})$$
$$C_{w_0 s_0 s_3 s_2} = t^{-9}(\hat{T} + \hat{T}_{s_0} + \hat{T}_{s_0 s_3} + \hat{T}_{s_0 s_3 s_2})$$
$$C_{w_0 s_0 s_1 s_3 s_2} = t^{-10}\left((1+2t^2)\hat{T} + (1+t^2)\hat{T}_{s_0} + \hat{T}_{s_0 s_1} + \hat{T}_{s_0 s_3} + \hat{T}_{s_0 s_1 s_3} + \hat{T}_{s_0 s_1 s_2} + \hat{T}_{s_0 s_3 s_2} + \hat{T}_{s_0 s_1 s_3 s_2}\right)$$

where $\hat{T}_w = \sum_{v \in W_0} T_{vw}$.

And then the KL-polynomials can be read off, since

$$C_w = t^{-\ell(w)} \sum P_{y,w} T_y .$$

We then read off each $\dim \operatorname{Ext}^n_{\mathcal{O}}(M(y\cdot -2\rho), L(w\cdot -2\rho))$ as the coefficient of $t^{\ell(w)-\ell(y)-n}$ in $P_{y,w}(t^2)$, and determine $\dim \operatorname{Ext}^n_{\mathcal{O}}(L(y\cdot -2\rho), L(w'\cdot -2\rho))$ as

$$\sum_{\substack{a+b=n \\ y}} \dim \operatorname{Ext}^a_{\mathcal{O}}(M(y\cdot -2\rho), L(w\cdot -2\rho)) \dim \operatorname{Ext}^b(M(y\cdot -2\rho), L(w'\cdot -2\rho)) .$$

We label the weights / Coxeter group elements as follows:

weight	elt.	label
$(0,0,0)$	w_0	1
$(2,0,2)$	$w_0 s_0$	2
$(3,1,1)$	$w_0 s_0 s_1$	3
$(1,1,3)$	$w_0 s_0 s_3$	4
$(2,2,2)$	$w_0 s_0 s_1 s_3$	5
$(5,0,1)$	$w_0 s_0 s_1 s_2$	6
$(1,0,5)$	$w_0 s_0 s_3 s_2$	7
$(3,2,3)$	$w_0 s_0 s_1 s_3 s_2$	8

From the form of the projective resolutions of simples & standard modules given by the $\mathrm{Ext}^{\bullet}(L,L)$ tables we can deduce the graded vector-space structure of the PIMs and standard modules, given overleaf.

PIMs

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 2 & & 5 & & \\ 1 & & 3 & & 4 & & 1 \\ & & & 2 & & & \end{array}$$

$$\begin{array}{ccccccccccccc} & & & & & & & & & & 2 & & \\ & & & & & 8 & & & 3 & & 1 & & 4 \\ 2 & & 6 & & 5 & & 7 & & 2 & & 5 & & 2 \\ & & 3 & & 4 & & & & 4 & & 1 & & 3 \\ & & & & & & & & & & 2 & & \end{array}$$

$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & 5 & & 2 & & 6 & & \\ 3 & & 1 & & 8 & & 4 & & 3 \\ 6 & & 2 & & 5 & & 2 & & 7 \\ & & & 3 & & 4 & & & \end{array}$$

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & 5 & & 2 & & 7 & & \\ 4 & & 1 & & 8 & & 3 & & 4 \\ 7 & & 2 & & 5 & & 2 & & 6 \\ & & & 4 & & 3 & & & \end{array}$$

$$\begin{array}{ccccccccc} & & & & 5 & & & & \\ & 3 & & 8 & & 1 & & 4 & \\ 6 & & 2 & & 5 & & 2 & & 7 \\ & & & 3 & & 4 & & & \end{array}$$

$$\begin{array}{ccccccc} & & 6 & & & & \\ & 3 & & & 8 & & \\ 5 & & 2 & & 6 & & 7 \\ & 3 & & 4 & & & \end{array}$$

$$\begin{array}{ccccccc} & & 7 & & & & \\ & 4 & & 8 & & & \\ 5 & & 2 & & 7 & & 6 \\ & & 4 & 3 & & & \end{array}$$

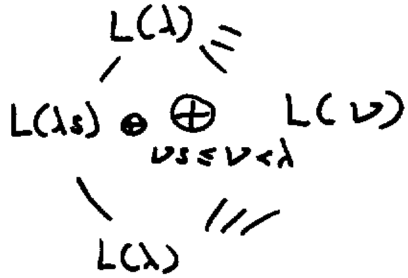
$$\begin{array}{ccccccc} & & & 8 & & & \\ 6 & & 5 & & 2 & & 7 \\ & & 3 & & 4 & & \end{array}$$

Standards

$$1 \qquad \begin{array}{c} 2 \\ 1 \end{array} \qquad \begin{array}{c} 3 \\ 2 \end{array} \qquad \begin{array}{c} 4 \\ 2 \end{array} \qquad \begin{array}{ccc} & 5 & \\ 3 & 1 & 4 \\ & 2 & \end{array} \qquad \begin{array}{ccccccc} & & & 8 & & & \\ 6 & & 5 & & 2 & & 7 \\ & & 3 & & 4 & & \end{array}$$

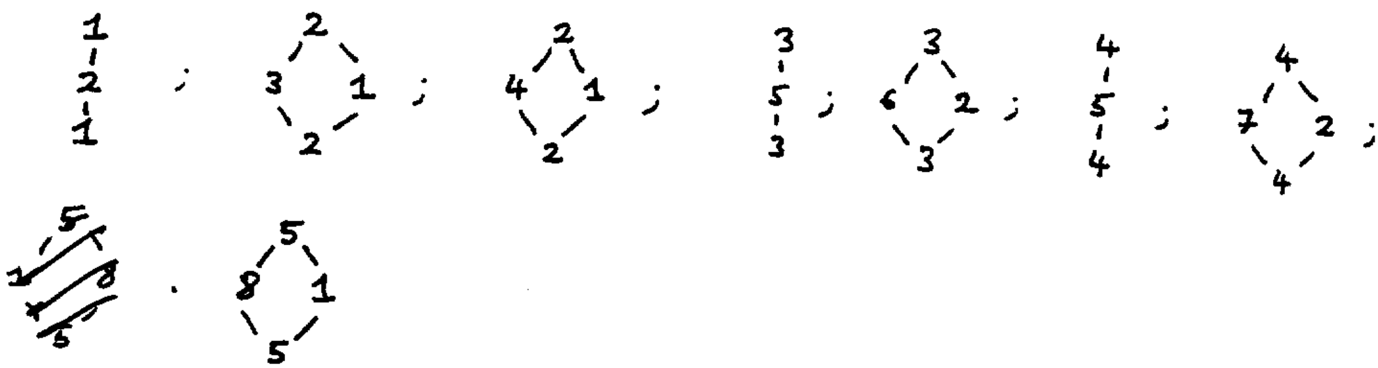
$$\begin{array}{c} 6 \\ 3 \end{array} \qquad \begin{array}{c} 7 \\ 4 \end{array}$$

Recall that, for any λ with $\lambda s > \lambda$, the module

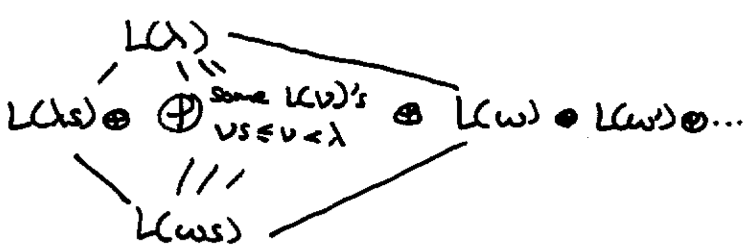


exists, as a quotient of $\frac{\Delta(\lambda)}{\Delta(\lambda s)}$, where $L(\nu)$ appears precisely the number of times it appears beneath the head of $\Delta(\lambda)$.

It follows that the following modules exist:



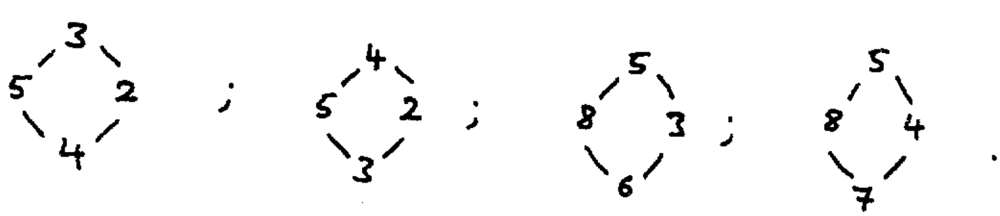
The following is also a quotient of $\frac{\Delta(\lambda)}{\Delta(\lambda s)}$:



Note: The existence of modules of this form is an assumption, based on a guess from the KL polynomial recursions. Possibly, the existence can be proved. --Len Scott

where $\omega s > \omega$

It follows that the following modules exist:



We will assume that the standard modules have the property that their radical series and socle series coincide. (0)
(This determines all standard modules except Δ_8).

<u>Key observation</u> There is an automorphism of W (given by conjugation by $-w_0$) which acts on W_0 and fixes s_0, s_2 and interchanges s_1, s_3. The corresponding action c on labels interchanges 3,4, interchanges 6,7 and fixes 1,2,5,8. Write Θ for the corresponding involution of the affine Lie algebra L (Cartan involution); then we have a functor f on the ~~category~~ subcategory of $\mathcal{O}_L$ consisting of modules composed entirely of the eight simples given, given by:

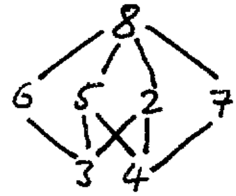
- $fM \cong M$ as a vector space
- for $y \in fM$ corresponding to $x \in M$, we have $a \cdot y = (\Theta a) \cdot x \in fM$ $(a \in L)$
- for $\alpha : M \to N$ we have $f\alpha(x) = \alpha x \in fN$

We can easily check that f maps simples to simples, projectives to projectives, PIMs to PIMs and $f^2 = \mathrm{id}$. Moreover f interchanges the simples labelled 3,4, interchanges the simples labelled 6,7 and fixes all other simples (up to isomorphism of course), and likewise for the PIMs. It acts on simple filtrations in the same way. (1)

Now write $P = P_1 \oplus \cdots \oplus P_8$ and observe that $fP \cong P$. Let us identify fP with P. Now ~~P will be~~ $\mathrm{End}_{\mathcal{O}_L} P$ will be Koszul, generated by idempotents $\iota_1, \ldots, \iota_8$ and homomorphisms ${}_ia_j$ mapping P_j to a degree 1 submodule of P_i, whenever $\dim \mathrm{Ext}_{\mathcal{O}_L}(L_i, L_j) = 1$. (and killing all ~~projectives~~ PIMs other than P_j).

Since f is a functor (and $f0 = 0$, $f1 = 1$) and additive, it follows that the induced map $f : \mathrm{End}_{\mathcal{O}_L} P \to \mathrm{End}_{\mathcal{O}_L} P$ is an algebra homomorphism (indeed isomorphism); further ~~$f\iota_1 = \iota_1, f\iota_2 = \iota_2, f\iota_3 = \iota_4, f\iota_4 = \iota_3, f\iota_5 = \iota_5, f\iota_6 = \iota_7, f\iota_7 = \iota_6, f\iota_8 = \iota_8$~~ $f(\iota_j) = \iota_{c(j)}$ for each j, and $f({}_ia_j) = {}_i\lambda_j \, {}_{c(i)}a_{c(j)}$ for each i, j with $\dim \mathrm{Ext}_{\mathcal{O}_L}(L_i, L_j) = 1$. (2)

Now since Δ_8 is a quotient of P_8 and they have equivalent simple filtrations, we have $\Delta_8 = P_8$. Combining (0) and (1) we determine the structure of P_8:



Now the PIMs have standard filtrations, and Δ_8 is the unique standard module containing the label 8, ~~so~~ and Δ_8 is projective; so it follows that every label 8 occurring in any PIM is at the head of a P_8 submodule.

~~In~~ We use the standard filtrations to make some progress on the structure of the PIMs. ~~[illegible]~~ The structure that they (initially) imply is drawn in blue ink; the notation is described below:

-  means that the copy of b shown is the unique degree 1 copy of b in the submodule generated by the copy of a shown

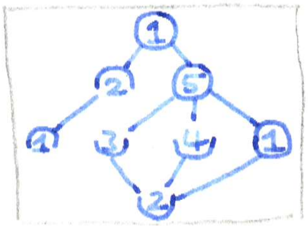
-  means that the unique copy of b described above is some strictly diagonal copy of b taken from the vector space generated by the copies of b shown

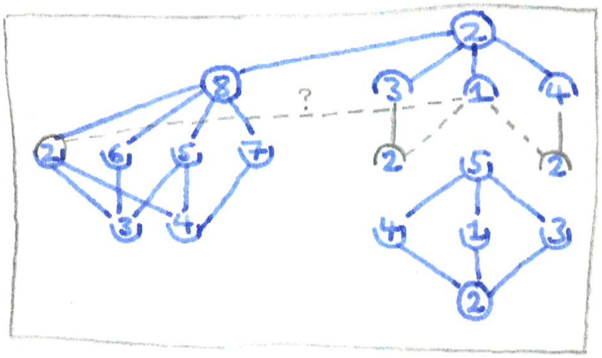
-  means that ~~all lines~~ no more lines may be drawn below the copy of a shown

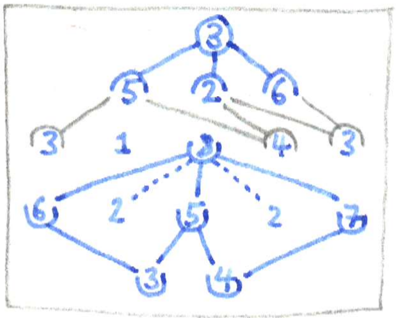
-  means that no more lines may be drawn above the copy of a shown.

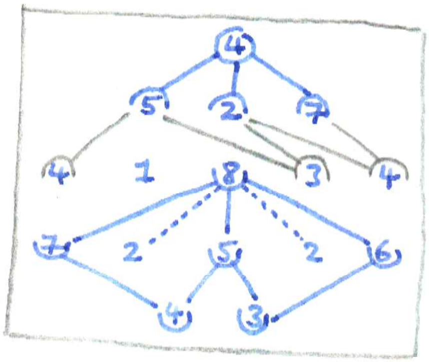
For reasons which will become clear later, ~~so~~ I have chosen to leave some ambiguity in the P_8-submodules of P_3, P_4, P_5.

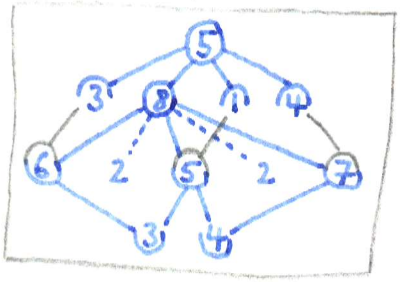
Next, I have drawn in pencil the structure implied by the existence of the modules on page 5.

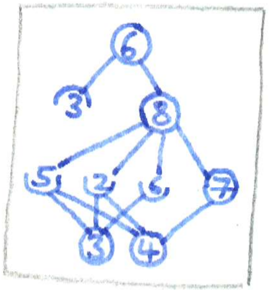


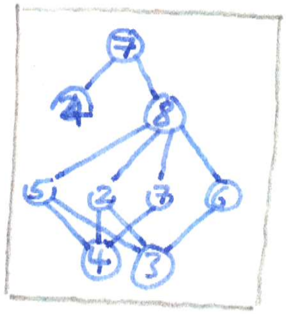


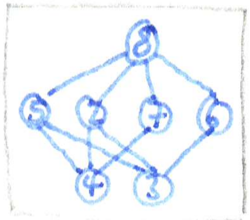




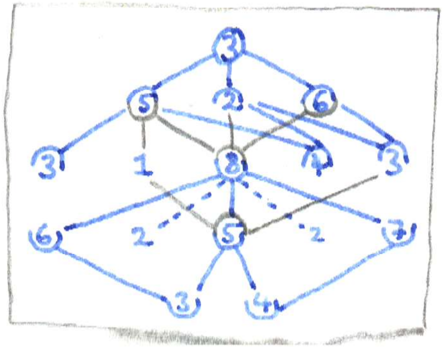


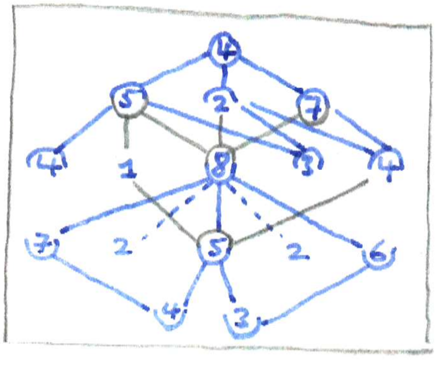




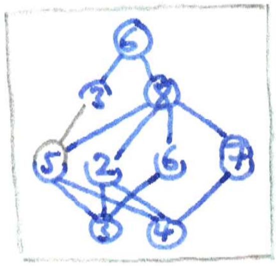


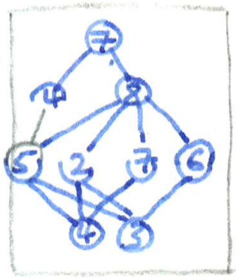
Next, we use duality to obtain information about P_3 and P_4 from P_5:



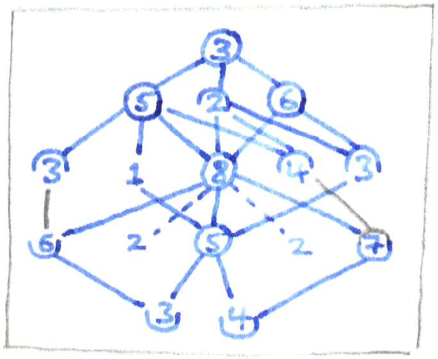


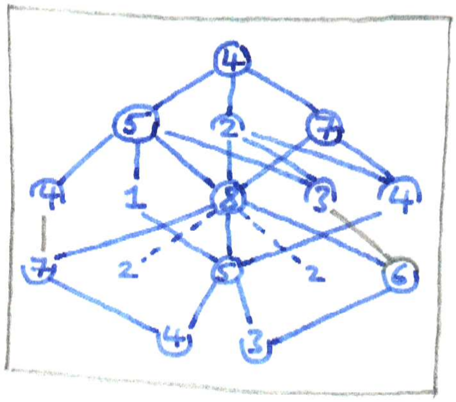
and also about P_6 and P_7:





Next we use duality to obtain information about P_3 and P_4 from P_3 and P_4:

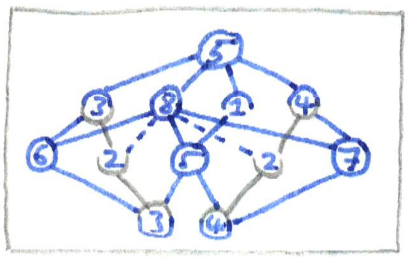




Note that this is (more than) enough to tell us that $P_5 \subseteq P_3$, $P_5 \subseteq P_4$, $P_6 \subseteq P_3$, $P_7 \subseteq P_4$.

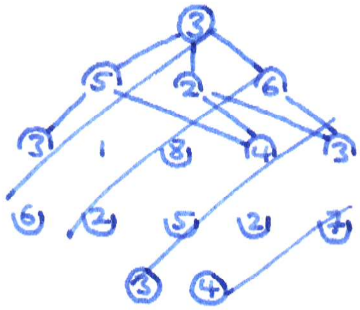
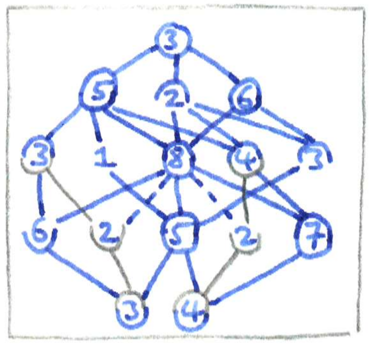
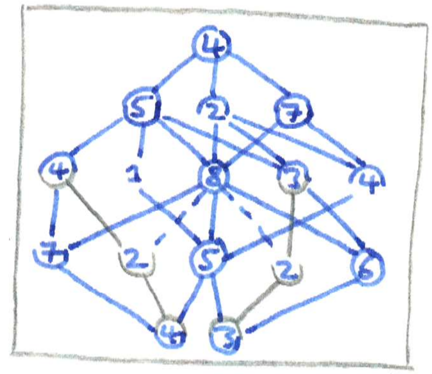
We will now use duality to obtain information about P_5 from P_3 and P_4.

Observe that the <u>chain</u> $\begin{array}{c}3\\ | \\ 2\\ | \\ 4\\ | \\ 5\end{array}$ does not exist in P_3, so $\begin{array}{c}5\\ | \\ 4\\ | \\ 2\\ | \\ 3\end{array}$ must not exist in P_5; likewise $\begin{array}{c}5\\ | \\ 3\\ | \\ 2\\ | \\ 4\end{array}$ also does not exist in P_5; however $\begin{array}{c}3\\ | \\ 2\\ | \\ 3\\ | \\ 5\end{array}$ does exist in P_3, so $\begin{array}{c}5\\ | \\ 3\\ | \\ 2\\ | \\ 3\end{array}$ does exist in P_5, and similarly so does $\begin{array}{c}5\\ | \\ 4\\ | \\ 2\\ | \\ 4\end{array}$. We thus update our diagram for P_5:

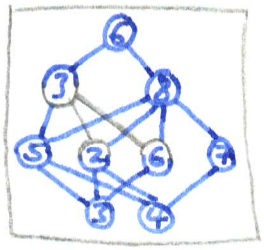
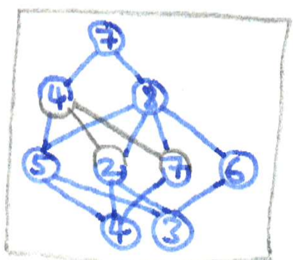


(with the 8 necessarily on top of a strict diagonal copy from the structure of P_8).

We can also update P_3, P_4 (since they contain P_5):

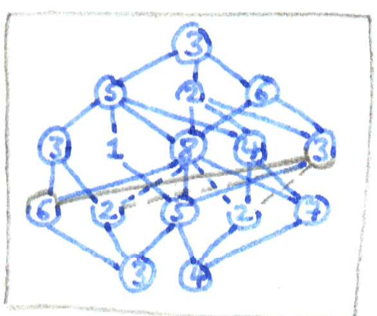
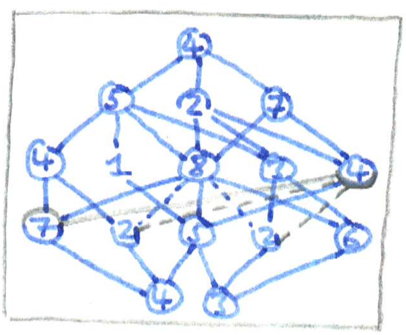




Next use duality to obtain information about P_6, P_7 from P_3, P_4:

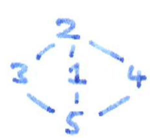
(Actually, duality gives us the chains $6 - 3 - 2 - 4$ & $7 - 4 - 2 - 3$; the chains $6 - 3 - 6 - 3$ & $7 - 4 - 7 - 4$ are given by the homomorphisms ${}_6a_3$, ${}_7a_4$).

Hence we update P_3, P_4:

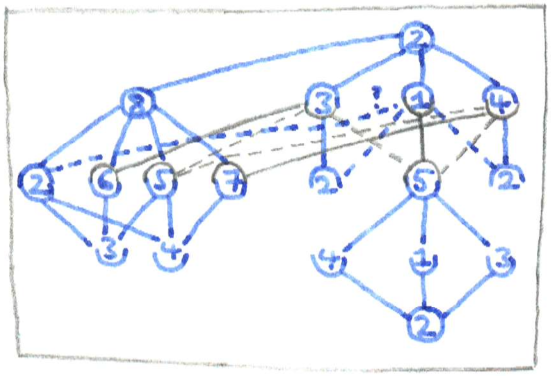



This shows that the chains $2 - 3 - 6$, $2 - 4 - 7$ exist in P_2.

Also, by duality we know

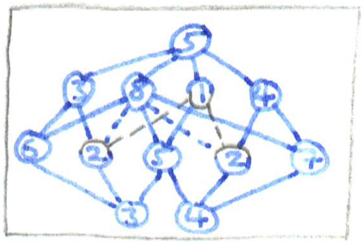


exists ; this allows us to update P_2 to the following.



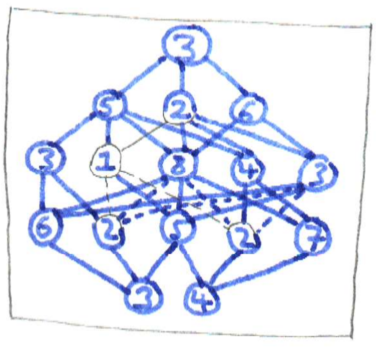
The reason that both 3 & 4 in the second row lie above strictly diagonal copies of the 5's as given in the third row is as follows: if not, then one of them (hence both) must lie above the 5 that is under the 1; this implies the existence of a module $\begin{smallmatrix}2\\8\\5\end{smallmatrix}$, hence of a module $\begin{smallmatrix}5\\8\\2\end{smallmatrix}$, which is absurd (look at P_5).

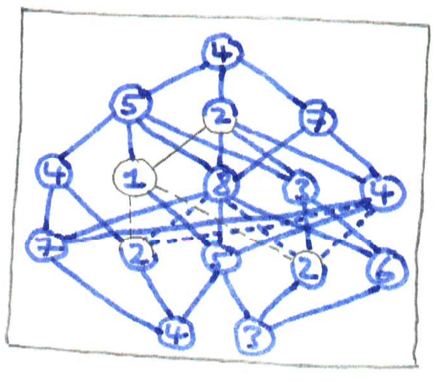
Using the structure of Δ_5 and ~~the~~ (1), we can complete the picture for P_5:



Note that the 1 and the 8 in the second row must be on top of distinct diagonal copies of 2 in the third row (else $\begin{smallmatrix} & 5 & \\ 3 & & 4 \\ & 2 & \end{smallmatrix}$ exists, hence $\begin{smallmatrix} & 2 & \\ 3 & & 4 \\ & 5 & \end{smallmatrix}$ exists, which is absurd (look at P_2)).

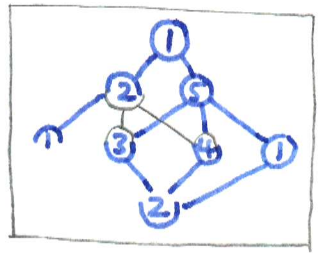
We can now complete the picture for P_3, P_4 (using ${}_3\Delta_5$, ${}_4\Delta_5$ and duality):





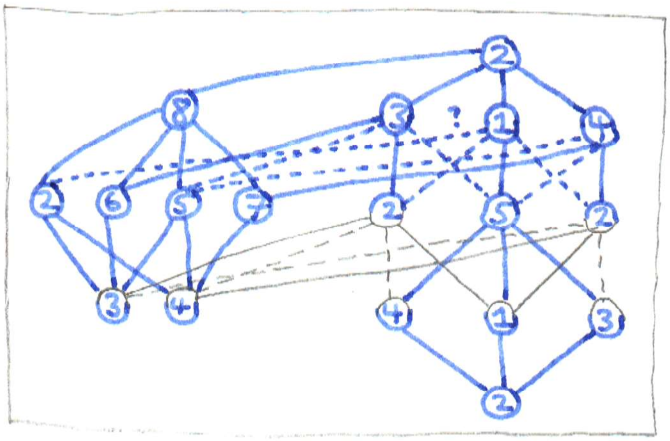
Bear in mind that in P_3, the chains $\begin{smallmatrix}3\\2\\7\\2\end{smallmatrix}$ and $\begin{smallmatrix}3\\2\\8\\2\end{smallmatrix}$ are on top of the same diagonal copy of label 2, and $\begin{smallmatrix}3\\2\\1\\2\end{smallmatrix}$ lies on top of a different diagonal copy of label 2 (and likewise for P_4).

${}_3a_1$ (or ${}_4a_1$) gives us the picture for P_1:

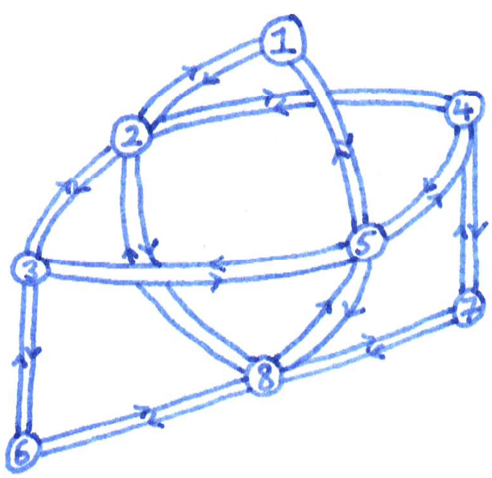


We won't be able to complete this picture until later.

${}_2a_3$, ${}_2a_4$ and duality (on P_1) allow us to complete the picture for P_2 (up to the existence of the dashed line marked with a "?"):



We will now compute quiver and relations for the endomorphism algebra. The quiver is drawn below:



We will write $i \cdots jkl$ for ${}_i a_{\cdot} \cdots {}_{\cdot} a_j \, {}_j a_k \, {}_k a_l$.

The following relations are automatic:

$858 = 0$
$868 = 0$
$878 = 0$
$828 = 0$
$158 = 0$
$128 = 0$
$535 = 0$
$545 = 0$
$821 = 0$
$851 = 0$.

We are free to insist (multiplying the $j a_i$ by appriate ~~scla~~ scalars) that:

$368 = 328 = 358$
$478 = 428 = 458$
$636 = 686$
$236 = 286$
$536 = 586$
$747 = 787$
$247 = 287$
$547 = 587$.

Now we know $363 = \lambda \cdot 323$ for some $\lambda \in \mathbb{C}^\times$.

Hence $0 \neq \lambda \cdot 3236 = 3636 = 3686 = 3286 = 3236$, so $\lambda = 1$ and $\underline{363 = 323}$.

Similarly $\underline{474 = 424}$.

We have also $453 = \lambda \cdot 423$ for some (new) $\lambda \in \mathbb{C}^\times$, so

$0 \neq \lambda \cdot 4236 = 4536 = 4586 = 4286 = 4236$, so $\lambda = 1$ and $\underline{453 = 423}$.

Similarly $\underline{354 = 324}$.

We are also free to insist that:

$635 = 685$ (as we've never used 85)
$515 = 585$ (as we've never used 51)
$632 = 682$ (as we've never used 82)
$123 = 153$ (as we've never used 12)

Let us now consider the group G of all diagonal changes of basis which preserve the insisted relations (hence also the implied relations).

The general element is given by:

$36 \longmapsto \alpha \cdot 36$
$68 \longmapsto \beta \cdot 68$
$32 \longmapsto \gamma \cdot 32$
$28 \longmapsto \alpha\beta\gamma^{-1} \cdot 28$
$35 \longmapsto \delta \cdot 35$
$58 \longmapsto \alpha\beta\delta^{-1} \cdot 58$
$47 \longmapsto \varepsilon \cdot 47$
$78 \longmapsto \zeta \cdot 78$
$42 \longmapsto \alpha^{-1}\beta^{-1}\gamma\varepsilon\zeta \cdot 42$
$45 \longmapsto \alpha^{-1}\beta^{-1}\delta\varepsilon\zeta \cdot 45$
$63 \longmapsto \eta \cdot 63$
$86 \longmapsto \alpha\beta^{-1}\eta \cdot 86$
$23 \longmapsto \alpha\gamma^{-1}\eta \cdot 23$
$53 \longmapsto \alpha\delta^{-1}\eta \cdot 53$
$74 \longmapsto \theta \cdot 74$
$87 \longmapsto \varepsilon\zeta^{-1}\theta \cdot 87$
$24 \longmapsto \alpha\beta\gamma^{-1}\zeta^{-1}\theta \cdot 24$
$54 \longmapsto \alpha\beta\delta^{-1}\zeta^{-1}\theta \cdot 54$
$85 \longmapsto \beta^{-1}\delta\eta \cdot 85$
$51 \longmapsto \iota \cdot 51$
$15 \longmapsto \alpha\eta\iota^{-1} \cdot 15$
$82 \longmapsto \beta^{-1}\gamma\eta \cdot 82$
$12 \longmapsto \alpha\gamma\delta^{-1}\eta\iota^{-1} \cdot 12$
$21 \longmapsto \kappa \cdot 21$

The general element maps

$745 \longmapsto \alpha^{-1}\beta^{-1}\delta\varepsilon\zeta\theta \cdot 745$

& $785 \longmapsto \beta^{-1}\delta\zeta\eta \cdot 785$

so we are free to insist that

$\underline{745 = 785}$

(by applying a certain element of G); the new relation-preserving group is obtained from G by insisting

$\underline{\theta = \alpha\varepsilon^{-1}\eta}$.

Now we know $285 = x \cdot 235 + y \cdot 245$ for some $x, y \in \mathbb{C}^\times$.

Hence $3285 = x \cdot 3235 + y \cdot 3245$.

But $3245 = 0$, so:

$x \cdot 3235 = 3285 = 3685 = 3635 = 3235$,

so $x = 1$. Similarly, $y = 1$ and

$\underline{285 = 235 + 245}$.

Now we use (2) (page 6).

We have $215 = x \cdot 235 + y \cdot 245$ with $x, y \in \mathbb{C}^\times$, $x \neq y$.

Applying f we have
$$ {}_2\lambda_1\,{}_1\lambda_5 \cdot 215 = {}_2\lambda_3\,{}_3\lambda_5\, x \cdot 245 + {}_2\lambda_4\,{}_4\lambda_5\, y \cdot 235 $$
$$ = {}_2\lambda_1\,{}_1\lambda_5\, x \cdot 235 + {}_2\lambda_1\,{}_1\lambda_5\, y \cdot 245 $$

and also
$$ {}_2\lambda_8\,{}_8\lambda_5 \cdot 285 = {}_2\lambda_3\,{}_3\lambda_5 \cdot 245 + {}_2\lambda_4\,{}_4\lambda_5 \cdot 235 $$
$$ = {}_2\lambda_8\,{}_8\lambda_5 \cdot 235 + {}_2\lambda_8\,{}_8\lambda_5 \cdot 245 $$

hence ${}_2\lambda_3\,{}_3\lambda_5 = {}_2\lambda_4\,{}_4\lambda_5$; ${}_2\lambda_3\,{}_3\lambda_5\, x = {}_2\lambda_1\,{}_1\lambda_5\, y$; ${}_2\lambda_4\,{}_4\lambda_5\, y = {}_2\lambda_1\,{}_1\lambda_5\, x$

so ${}_2\lambda_3\,{}_3\lambda_5\, x^2 = {}_2\lambda_4\,{}_4\lambda_5\, y^2 = {}_2\lambda_3\,{}_3\lambda_5\, y^2$, hence $x^2 = y^2$ so $x = -y$.

We may assume $x = 1$, so that $\underline{215 = 235 - 245}$

~~assumes~~ and we must restrict $\underline{\kappa = \gamma'\delta\iota}$.

Now $321 = \lambda . 351$, so

$0 \neq \lambda . 3515 = 3215 = 3235 - 3\overset{0}{\cancel{2}}45 = 3635 = 3685 = 3585 = 3515$, so $\underline{321 = 351}$

Similarly $\underline{421 = -451}$.

Also $823 = \lambda . 863$, so

$0 \neq \lambda . 6863 = 6823 = 6323 = 6363 = 6863$, so $\underline{823 = 863}$

and $824 = \lambda . 854$, so

$0 \neq \lambda . 6854 = 6824 = 6324 = 6354 = 6854$, so $\underline{824 = 854}$.

At this stage, it can be checked that each $\phi \in G$ maps the element $i \cdots jkl$ to $x . i \cdots jkl$ for some x depending only on ϕ, i and l (x defined for $i \cdots jkl \neq 0$). So G ~~acts as~~ consists of all diagonal automorphisms of A ($= \mathrm{End}_G P$) (i.e. diagonally on the given basis of A_1).

Let g be the linear map on A_1 which maps ${}_i a_j$ to ${}_{c(i)} a_{c(j)}$, extended to TA_1.

Consider gf. It acts diagonally on A_1, mapping ${}_i a_j$ to ${}_i \lambda_j \, {}_i a_j$.

It "preserves most relations" – that is, most of the quadratic relations $X = 0$ ($X \in T^2 A_1$) that we've insisted on have $gf X \propto X$. To see this, observe (for instance) that ~~since~~ $368 - 328 = 0$ and $478 - 428 = 0$, and f is an automorphism of A, so $f . (368 - 328) \propto 478 - 428$, whence $gf . (368 - 328) \propto 368 - 328$.

The only insisted quadratic relations which are not (necessarily) preserved by gf are: $632 - 682$

$123 - 153$

$215 - 235 + 245$ (instead $gf (215 - 235 + 245) \propto 215 + 235 - 245$).

Since $632 = 682$, $123 = 153$ are the only insisted relations involving 82, 12 respectively, ~~so~~ there exist $\alpha, \beta \in \mathbb{C}^\times$ such that the diagonal map h ~~sends~~ sending $21 \mapsto -21$, $12 \mapsto \alpha . 12$ & $82 \mapsto \beta . 82$ and fixing all other elements of the basis for A_1 has the same effect (up to scalar) on the insisted relations as gf. Hence $hG = gfG$,

so gh is an automorphism of A.

This allows us to write down all remaining (quadratic) relations:

$$\left.\begin{aligned} 742 &= \beta \cdot 782 \\ 154 &= \alpha \cdot 124 \\ 824 = 854 &= \beta^{-1} \cdot 874 \\ 823 = 863 &= \beta^{-1} \cdot 853 \end{aligned}\right\} \text{applying } gh \text{ to } \begin{aligned} 0 &= 632 - 682 \\ &= 153 - 123 \\ &= 823 - 863 \\ &= 824 - 854 \end{aligned}$$

$532 = n \cdot 582 + m \cdot 512$ (some $n, m \in \mathbb{C}^\times$)

so $542 = \beta n \cdot 582 + \alpha m \cdot 512$ (applying gh)

so $\alpha^2 = 1 = \beta^2$ (applying gh again)

so $\{\alpha, \beta\} = \{1, -1\}$ (since 542, 532 are not proportional)

so $\alpha = -\beta$ and $\beta = \pm 1$.

Hence:

$$\begin{aligned} 742 &= \beta \cdot 782 \\ 154 &= -\beta \cdot 124 \\ 824 = 854 &= \beta \cdot 874 \\ 823 = 863 &= \beta \cdot 853 \\ 532 &= n \cdot 582 + m \cdot 512 \\ 542 &= \beta(n \cdot 582 - m \cdot 512) \\ 212 &= p \cdot 232 + \beta p \cdot 242 + q \cdot 282 \qquad \text{some } p \in \mathbb{C}^\times,\ q \in \mathbb{C}. \end{aligned}$$

As far as I can tell, there is no way to rule out one of the two cases using only KL theory. However the constants n, m, p, q can be determined in either case.

<u>Case $\beta = 1$</u>.

$$\begin{aligned} 21532 &= m \cdot 21512 = -21542 \\ &= m \cdot (23512 - 24512) \\ &= m \cdot (23212 + 24212) \\ &= mp \cdot (23232 + 23242 + 24232 + 24242) \\ &= mp \cdot (23542 + 24532) \\ &= \tfrac{1}{2} mp \cdot (21542 - 21532) \\ &= -mp \cdot 21532 \neq 0 \end{aligned}$$

$\therefore \boxed{mp = -1}$

$$\begin{aligned} 5323 = 5363 = 5863 &= 5823 \\ &= 5853 = 5153 = 5123 \\ &= n \cdot 5823 + m \cdot 5123 = (n+m) \cdot 5323 \neq 0 \end{aligned}$$

$\therefore \boxed{n+m = 1}$

$1232 = 1532 = m \cdot 1512$ | $1242 = -1542 = m \cdot 1512 = 1232$

$\therefore\ p \cdot 1232 = -1512$

$= 1212 - p \cdot 1242$

So (crucially) $1212 = 2p \cdot 1232 \neq 0$.

Next. $p \cdot 1242 = 1212 + 1512$

$$\therefore\ p \cdot 21242 = 21212 + 21512 = 21212 + \overset{23512}{\cancel{23542}} - \overset{24512}{\cancel{21542}}$$

$$= 21212 + 23212 + 24212$$

$$\therefore\ p^2 \cdot 21242 = p \cdot 21212 + 21212$$

$$\therefore\ p^2 \cdot (21242 + 21232) = (2p+2) \cdot 21212$$

$$= p \cdot 21212 \quad \neq 0 \quad \text{so} \quad \underline{p = -2}$$

$$\underline{m = 1/2}$$

$$\underline{n = 1/2}.$$

Finally, $3212 = -2 \cdot 3232 - 2 \cdot 3242 + q \cdot 3282$

$$= -2 \cdot 3632 - 2 \cdot 3542 + q \cdot 3682$$

$$= -2 \cdot 3682 + q \cdot 3682 - 3582 + 3512$$

$$= (q-3) \cdot 3682 + 3212$$

$$\therefore\ (q-3) \cdot 3682 = 0$$

so $\underline{q = 3}$ $\quad (3682 \neq 0)$

Case $\underline{\beta = -1}$ We introduce new constants to be determined (for clarity).

We can write:

$$\left.\begin{aligned} 512 &= x \cdot (532 + 542) \\ 582 &= y \cdot (532 - 542) \\ 212 &= r \cdot (232 - 242) + s \cdot 282 \end{aligned}\right\} \begin{aligned} x, y, r &\in \mathbb{C}^\times \\ s &\in \mathbb{C} \end{aligned}$$

In this case, $1212 = r \cdot (1232 - 1242) = r \cdot (1532 - 1542)$

$= r y^{-1} \cdot 1582 = 0$. $\quad$ (so $1532 = 1542$)

We have:

$$21512 = x \cdot (21532 + 21542) = 2x \cdot 21532 = 2x \cdot 21542$$

$$= 23512 - 24512$$

$$= 23212 + 24212$$

$$= r \cdot (23\cancel{2}32 - 23242 + 24232 - 24\cancel{2}42)$$

$$= r \cdot (24532 - 23542)$$

$$= -\tfrac{1}{2} r \cdot (21532 + 21542) = -r \cdot 21532 \quad \therefore \quad \boxed{r = -2x}$$

$5323 = 5363 = 5863 = 5823$

$= -5853 = -5153 = -5123$

$y \cdot 5123 + x \cdot 5823 = 2xy \cdot 5323$

$= (x-y) \cdot 5323 \quad \neq 0 \quad \therefore \quad \boxed{2xy = x-y}$

$3282 = 3682 = 3632 = 3232$

$\therefore \; s \cdot 3282 = s \cdot 3232$

$= 3212 - r \cdot (3232 - 3242)$

$\therefore \; (r+s) \cdot 3282 = 3212 + r \cdot 3242$

$\therefore \; (r+s) \cdot 32823 = 32123 = 32153 = 32353 = 36353 = 36853 = 32853$

$= -32823 \neq 0 \quad \therefore \quad \boxed{r+s=-1}$

$-3282 = 3212 + r \cdot 3242$

$\therefore \; -32824 = 32124 + r \cdot 32424.$

$32124 = 32154 = 32354 = 36354 = 36854 = 32824$

$\therefore \; r \cdot 32424 = -2 \cdot 32824 \quad \neq 0$

$32424 = 32474 = 32874 = -32824$

$\therefore \; \underline{r=2}$

$\underline{s=-3}$

$\underline{x=-1}$

$\underline{y=1}$

By reintroducing β we can list all (quadratic) relations, ~~shown on the next page~~ for both cases simultaneously, shown on the next page:

$858 = 0$ $\qquad \beta = \pm 1$.

$868 = 0$

$878 = 0$

$828 = 0$

$158 = 0$

$128 = 0$

$368 = 328 = 358$

$478 = 428 = 458$

$636 = 686$

$236 = 286$

$536 = 586$

$747 = 787$

$247 = 287$

$547 = 587$

$535 = 0$

$545 = 0$

$515 = 585$

$635 = 685$

$745 = 785$

$285 = 235 + 245$

$215 = 235 - 245$

$363 = 323$

$453 = 423$

$123 = 153$

$863 = 823 = \beta \cdot 853$

$474 = 424$

$354 = 324$

$124 = -\beta \cdot 154$

$824 = 854 = \beta \cdot 874$

$632 = 682$

$742 = \beta \cdot 782$

$512 = \beta \cdot 532 - 542$

$582 = 532 + \beta \cdot 542$

$212 = -2\beta \cdot 232 - 2 \cdot 242 + 3\beta \cdot 282$

$821 = 0$

$851 = 0$

$321 = 351$

$421 = -451$.