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Isomorphisms of p-adic group rings.

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The famous isomorphism problem of group rings proposed by G. Higman in his 1940 Ph.D. thesis [see Proc. London Math. Soc. (2) 46 (1940), 231–248; MR 2, 5] and R. Brauer [Lectures on modern mathematics, Vol. 1, 133–175, Wiley, New York, 1963; MR 31 #2314] asks: Given finite groups G and H, does the isomorphism of the integral group rings $\mathbf{Z}G$ and $\mathbf{Z}H$ imply the isomorphism of the groups G and H?

H. Zassenhaus [see Sudarshan K. Sehgal, Methods in ring theory (Antwerp, 1983), 497–504, Reidel, Dordrecht, 1984; MR 86d:16012] strengthened this to the following conjecture: (ZC2) $\mathbf{Z}G = \mathbf{Z}H \Rightarrow H = \pm\alpha^{-1}G\alpha = \pm G^\alpha$ for some $\alpha \in \mathbf{Q}G$, the rational group algebra.

In this paper the authors use ingenious and quite difficult methods to prove the above conjecture for nilpotent groups (in fact, the coefficient rings are allowed to be more general). Moreover, for $\mathbf{Z}_pG$, the group ring of a p-group G over the p-adic integers $\mathbf{Z}_p$, the authors prove a very strong version of (ZC2), namely, $\mathbf{Z}_pG = \mathbf{Z}_pH$ and G a p-group $\Rightarrow H = G^\alpha$ for some $\alpha \in \mathbf{Z}_pG$. Amazingly, in this case, it is possible to choose $\alpha \in \mathbf{Z}_pG$ itself. It is not necessary to go to the quotient algebra. This version lends itself to induction. The authors are to be congratulated on this breakthrough.

In addition to (ZC2) above, Zassenhaus also conjectured: (ZC1) u a torsion unit in $\mathbf{Z}G \Rightarrow u = \pm g^\alpha$ for some $g \in G$, $\alpha \in \mathbf{Q}G$. Both (ZC1) and (ZC2) are special cases of a third conjecture of Zassenhaus: (ZC3) Let H be any finite subgroup of the unit group of $\mathbf{Z}G$. Then there exists an $\alpha \in \mathbf{Q}G$ such that H^α is contained in $\pm G$.

The authors prove in a footnote that they are able to prove (ZC3) for $p = 2$ in $\mathbf{Z}_pG$, G a p-group. A. Weiss [Ann. of Math. (2) 127 (1988), no. 2, 317–332] has proved (ZC3) for all p. There is not too much known regarding these conjectures beyond nilpotent groups. (ZC3) is not yet known even for nilpotent groups. (ZC1) is known to be true for metacyclic groups $\langle a\rangle \ltimes \langle b\rangle$ with $(|a|, |b|) = 1$ [C. Polcino Milies, J. Ritter and Sudarshan K. Sehgal, Proc. Amer. Math.

Soc. 97 (1986), no. 2, 201–206; MR 87i:16013], for split metabelian groups with faithful action [Sudarshan K. Sehgal and Weiss, J. Algebra 103 (1986), no. 2, 490–499; MR 88f:20015], and for some other metabelian groups [Z. Marciniak, Ritter, Sudarshan K. Sehgal and Weiss, J. Number Theory 25 (1987), no. 3, 340–352; MR 88k:20019]. (ZC2) is known to be true for abelian by nilpotent class two groups, with relatively prime orders [Sudarshan K. Sehgal, the reviewer and Zassenhaus, Comm. Algebra 12 (1984), no. 19-20, 2401–2407; MR 85m:20008] and for some metabelian groups [G. L. Peterson, Illinois J. Math. 21 (1977), no. 4, 836–844; MR 57 #3184b].

This latest advance should spark more activity on these problems. This paper also contains other interesting results.

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