Are All Groups Finite?

*Leonard L. Scott**

This paper is dedicated to Walter Feit on the occasion of his 65th birthday. Its contents were presented in part at the 1995 Ohio State finite group representation conference organized in celebration of that birthday. Primarily, the paper is a discussion of some classical and recent developments in the modular representation theory of finite groups of Lie type, and the problems which drive that theory. But there is also a philosophical thread ...

An old question which arose again at the conference is the following: Are all groups finite? That is, applications and broader issues aside, if we think only of our interest in finite group theory itself, is it possible to safely ignore other groups? My viewpoint is that the answer to this question has two parts: First, in representation theory, at least, we cannot ignore the infinite complex Lie groups and their characteristic p analogs, the algebraic groups over $\bar{\mathbb{F}}_p$. The second part of my answer is that we can, nevertheless, hope to find understandings within finite group theory and finite dimensional algebra of ideas naturally suggested by these continuous contexts, and take them further.

Let me begin by convincing you of the first part of my answer: Suppose one is considering a finite group $G(\mathbb{F}_q)$ of Lie type, such as the special linear group $SL(n,q)$ of degree n with coefficients in the field $\mathbb{F}_q$ of q elements, q a power of a prime p. The Classification of finite simple groups asserts that almost all of the latter are variations on the finite groups of Lie type together with the alternating groups. Much earlier (1963), Steinberg **[34]**, **[35]** proved all irreducible representations of $G(\mathbb{F}_q)$ with coefficients in a finite field of characteristic p, and, thus, in the algebraic closure $\bar{\mathbb{F}}_q$, come by restriction from the irreducible representations over $\bar{\mathbb{F}}_q$ of the algebraic group $G(\bar{\mathbb{F}}_q)$. The latter group is, of course, quite infinite. It is the analog via the Zariski topology, of the complex analytic Lie group $G(\mathbb{C})$. Moreover, the representations we need are continuous, and even "analytic", in the sense that they are locally defined by polynomial functions. Now, the theory of finite-dimensional

* The author thanks NSF for its support.

Offprint from: Representation Theory of Finite Groups, Ed.: R. Solomon

irreducible continuous representations of $G(\mathbb{C})$ has an elegant and powerful formulation, first, in that the irreducible representations are parametrized very completely by the "theory of the highest weight" of Cartan, and, second, that the characters of these representations are known, given by the famous "Weyl character formula". See, for instance, **[20]** for these theories for complex semisimple Lie algebras.

If we had such a parameterization and such a character formula for the finite groups $G(\mathbb{F}_q)$, not only would we would know their irreducible characters in the describing characteristic p of $G(\mathbb{F}_q)$, but we might learn something about the nondescribing case as well, where many analogies with the describing case have been discovered by Dipper and James **[15]**, **[16]**, **[14]**, **[21]**. Working with the general linear group $GL(n, q)$ they have found families of finite-dimensional algebras, the q-Schur algebras, parameterized by a variable q, that control the nondescribing characteristic representation theory of the group $GL(n, q)$ when q is taken to be a prime power (which may also be viewed as a kind of root of unity when the underlying field has positive characteristic), and which control the modular theory in the describing characteristic p when $q = 1$. If we knew both describing and nondescribing modular theory for the simple or nearly simple groups of Lie type, we could also hope to learn much about the maximal subgroup structure of all other finite groups **[30]**, **[2]**. Indeed, this is a main organizational theme of the upcoming 1997 Newton Institute program at Cambridge.

So, to summarize, it would be highly desirable to have for finite groups $G(\mathbb{F}_q)$ of Lie type a parameterization and character formula, as exist for the complex Lie groups $G(\mathbb{C})$. Also, thanks to the work of Steinberg mentioned above, both issues for $G(\mathbb{F}_q)$ reduce to the corresponding problem for the algebraic group $G(\bar{\mathbb{F}}_q)$. Now it is time to tell you that the parameterization problem for $G(\bar{\mathbb{F}}_q)$ was solved even before Steinberg's work by Chevalley, imitating the Lie-theoretic case $G(\mathbb{C})$ mentioned above.

Before discussing the character formula issue, let's consider how far we have come in discussing my reply to the philosophical issue, "Are all groups finite?" The first suggestion in my reply is that we cannot ignore $G(\mathbb{C})$ and $G(\bar{\mathbb{F}}_q)$, and I hope the initial history above of the parameterization for describing characteristic representations of $G(\mathbb{F}_q)$, through Steinberg and Chevalley, duplicating Cartan's "theory of the highest weight", is convincing evidence of the usefulness of looking at continuous and algebraic groups. The problems are easier for these more richly structured groups, and some have been solved. The second suggestion in my reply, that we can abstract from these continuous contexts, and perhaps go beyond them, is evidenced by what has happened to the parameterization theory since that time: First, Curtis and Richen **[12]**, **[13]**, **[29]** showed it was possible to carry out an analog of the parameterization process, suitably modified, in any finite group with an appropriate split BN pair. Second, Alperin **[1]** demonstrated with his celebrated conjecture (a main theme of our conference) that it was possible to formulate a version of the parameterization which makes sense for any finite group!

Perhaps those involved in the Classification might also say that it was useful to know that corresponding simple group classifications already existed for $G(\mathbb{C})$ and $G(\bar{\mathbb{F}}_q)$, and the uniform theory of groups of Lie type which emerged was (and is) useful in efficiently dealing with many properties of known groups, and in formulating many general concepts. The structure in the Lie-theoretic case was not at all ignored, but, as above, it was only a starting point (together with involutions and the Odd Order Paper!) for a more general (and more elaborate) theory.

Let's now go to the issue of a character formula for the describing characteristic representations of $G(\mathbb{F}_q)$. We are, I believe, far from a result as complete as the Classification, for irreducible modular representations of $G(\mathbb{F}_q)$. The few results we have put us at the beginning of the cycle, where it is still essential to learn from $G(\mathbb{C})$ and $G(\bar{\mathbb{F}}_q)$. Nevertheless, it has been part of the point of view of myself and my colleagues, especially in CPS (Ed Cline, Brian Parshall, and myself), to develop a theory as purely algebraic as possible, to both try and attain a more general theory and to allow for elaborations diverging from the cleanest cases. This point of view is also important in our approach to proving the current main conjecture, due to Lusztig. Before describing it, let me describe one of CPS's main algebraic abstractions, which will at least make the Lusztig conjecture easier to explain. The discussion is largely borrowed from my exposition **[32]**.

1.1 Highest weight categories, and examples. Fix a field k, and let $\mathcal{C}$ be an abelian k-category (that is, $\mathcal{C}$ is abelian, all Hom sets are k-modules, and multiplications of morphisms is k-linear). In all cases we will consider here, $\mathcal{C}$ will simply be equivalent to the category of finite dimensional modules over a finite dimensional algebra, but it may not start out looking like that. We suppose the nonisomorphic irreducible objects $L(\lambda)$ to be indexed by the elements λ of a poset Λ, called *weights*. For simplicity we will assume Λ is finite here; for a more general notion (requiring only that the *intervals* of Λ be finite), the reader is referred to **[7]**. We will also assume for simplicity that all objects of $\mathcal{C}$ have finite length, that Hom sets between objects are finite-dimensional over k, and, moreover, that $\mathrm{End}_C\, L(\lambda) \cong k$ for each $\lambda \in \Lambda$. We assume that $\mathcal{C}$ has enough projectives, and let $P(\lambda)$ denote the projective cover of $L(\lambda)$. We say that $\mathcal{C}$ is a *highest weight* category if there are objects $V(\lambda), \lambda \in \Lambda$, such that

(1) $V(\lambda)$ has head $L(\lambda)$, and all other composition factors of $V(\lambda)$ have smaller weight than λ.

(2) There is an epimorphism $P(\lambda) \to V(\lambda)$ with kernel filtered by objects $V(\mu)$ with μ greater than λ.

These conditions imply that $V(\lambda)$ is the largest epimorphic image of $P(\lambda)$ with λ maximal among the weights of its composition factors. Such objects arise naturally in Lie-theoretic contexts. We call $V(\lambda)$ a *Weyl object*, since it is a Weyl module in our favorite context of characteristic p algebraic group representations. Other good names are *Verma object*, or simply *standard object*. Typically, these objects are well understood, and the main object of research is to write the irreducible objects in terms of them in the Grothendieck group (that is, to obtain their "Weyl character formula"). This is precisely what the Lusztig conjecture purports to do, with certain restrictions.

Three examples in Lie theory.

(1) The example which first motivated CPS is the following: Let $G = G(k)$ be a semisimple, simply connected algebraic group over an algebraically closed field k of positive characteristic p (e.g. $k = \bar{\mathbb{F}}_q$). This is the most relevant example for finite group theory. In discussing it, I will assume some basic terminology from algebraic group theory, but the reader familiar with the basic theory of root systems and Lie algebras as found in **[20]** should be able to follow much of it. Just keep in mind the basic example $G = SL(n, \bar{\mathbb{F}}_q)$. There, T below is the group of invertible diagonal matrices over $\bar{\mathbb{F}}_q$, B is the group of upper triangular matrices and W is the group of $n \times n$ permutation matrices. The Lie algebra of G as a vector space is $n \times n$ matrices of trace 0, and its root spaces are just the $1-$dimensional spaces with arbitrary entries in the i, j position, for fixed $i \neq j$, and 0's elsewhere. These are common eigenspaces for the action of T by conjugation, and the associated homomorphism (character) mapping T to $\bar{\mathbb{F}}_q^\times$ is called a *root* in the world of algebraic groups, while an arbitrary algebraic group homomorphism from T to $\bar{\mathbb{F}}_q$ is called a *weight*. Let T be a fixed maximal torus, and denote the root system of T acting on the Lie algebra of G by Φ. We choose a

set Φ^+ of positive roots, and let B denote the corresponding Borel subgroup corresponding to the associated set Φ^- of *negative roots.* The set $X(T)$ of characters (weights) on T is partially ordered by the rule: $\lambda \leq \mu \Leftrightarrow \mu - \lambda = \sum_{\alpha \in \Phi^+} n_\alpha \alpha$ for non-negative integers n_α. We also have an induced poset structure on the set $X(T)^+$ of dominant weights (relative to Φ^+). Fix any finite set Λ_0 of dominant weights, let Λ be the (finite) set of dominant weights λ for which $\lambda \leq \lambda_0$ for some $\lambda_0 \in \Lambda_0$. Then the category $\mathcal{C}$ of finite-dimensional $G-$ modules (in the sense of algebraic groups) which have composition factors each with maximal $T-$ weight in Λ is a highest weight category with weight poset Λ. The Weyl modules $V(\lambda)$ are obtained as linear duals of modules induced to G, in the sense of algebraic groups, from dominant weights in $X(T)^+$ extended to B. (These induced modules are all finite-dimensional!) They may also be obtained by a reduction modulo p process from an irreducible module in characteristic 0. As such, their decomposition into weights for T is directly obtainable from the Weyl character formula.

Projecting $\mathcal{C}$ onto any block of G-modules also gives a highest weight category. If $p \geq h$, the Coxeter number of the root system, it is well-known **[22]** that the character formulas for all irreducible modules are deducible from those in the principal block. The weights for the latter are the dominant weights in the orbit $W_p.\,0$ of 0 under the 'dot' action of the affine Weyl group W_p (defined by $w.\,\mu = w(\mu + \rho) - \rho$, where ρ is the sum of all the fundamental dominant weights, for $w \in W_p$ and $\mu \in X(T)$.) Also, Steinberg's tensor product theorem allows us to restrict attention to *restricted* weights, those with coefficients less than p when expressed in terms of certain 'fundamental' weights. Let us redefine Λ as the set of dominant weights which are in the orbit $W_p.0$ and bounded above by a restricted weight in that orbit. Lusztig's conjecture may then be written

$$\operatorname{ch} L(w.0) = \sum_{y.0 \in \Lambda} (-1)^{\ell(w)-\ell(y)} P_{yw_0, ww_0}(1) \operatorname{ch} V(y.0),$$

for any weight $w.0$ in Λ. Here y, w are in W_p and w_0 denotes the long word in the ordinary Weyl group W. The terms $P_{yw_0,ww_0}(1)$, are values at 1 of *Kazhdan–Lusztig polynomials*, which are defined in a purely combinatorial way for any pair of elements in a Coxeter group. Finally $\ell(w)$ denotes the length of w in the sense of Coxeter groups (the number of fundamental reflections in a minimal expression), and the function ch$(-)$ just assigns an object of $\mathcal{C}$ to the associated element in the Grothendieck group of $\mathcal{C}$. The conjectured formula would give character formulas for all irreducible G-modules, so long as $p \geq 2h-3$, and these would in turn give corresponding character formulas for any finite group $G(\mathbb{F}_q)$ of Lie type associated to G, with q a power of p. (Actually, so long as $p \geq h$, the above formula could hold for all restricted weights, and as such would have the same implications for finite groups, for such a p. This stronger version of Lusztig's conjecture was formulated by Kato [23], who apparently originally believed it to be a consequence of the original conjecture, but the arithmetic doesn't work out that way. Let me take this opportunity to mention that the first open cases for the Lusztig conjecture occur for $SL(5,5)$ and $SL(5,7)$, which I have been examining with the help of an NSF undergraduate REU student. The former case is the first possibility for the Kato and Lusztig versions to diverge.)

Lusztig obtained his conjecture by analogy with his conjecture with Kazhdan **[24]** for complex Lie algebras, which we describe next.

(2) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and fix a Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Consider the corresponding category $\mathcal{O}$ of BGG. The objects are the $\mathfrak{g}$-modules which are $\mathfrak{h}$-diagonalizable with finite-dimensional weight spaces (where a 'weight' here is a 1-dimensional representation for the Lie algebra $\mathfrak{h}$), and with the set of nonzero weights bounded above by some finite set of weights. We will also restrict attention to the case where all weights are *integral*; equivalently, they belong (by identification) to the set $X(T)$ of characters for a torus T associated to $\mathfrak{h}$; these are just the integral linear combinations of the 'fundamental' weights for $\mathfrak{h}$. It is again true that any block of such modules forms a highest weight category, and all character formulas for irreducible modules are obtainable from the principal block case. The standard objects this time are the Verma modules $M_\lambda, \lambda \in X(T)$, obtained by tensor induction of λ at the universal enveloping algebra level from $\mathfrak{b}$ to $\mathfrak{g}$. We write $V(\lambda) = M_\lambda$, and let $L(\lambda)$ denote the irreducible head of $V(\lambda)$. The weights Λ indexing irreducible modules in the principal block are just those in the orbit $\Lambda = W. -2\rho$. (This is also the orbit of 0 under the 'dot' action, since $0 = w_0. -2\rho$.) They correspond bijectively to elements of the Weyl group. The Kazhdan–Lusztig conjecture (now a theorem due to Brylinski–Kashiwara **[6]** and Beilinson–Bernstein **[5]**) reads

$$\operatorname{ch} L(w. - 2\rho) = \sum_{y \in W} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch} V(y. - 2\rho),$$

where again $P_{y,w}(1)$ is the value at 1 of a Kazhdan–Lusztig polynomial.

The similarity of this formula and the previous one is remarkable, and all the more so when one considers that the standard modules in the first case are finite-dimensional, but infinite-dimensional here. The next case, is even more remarkable, in that we obtain precisely the same character formula for standard objects which are not modules at all, but complexes of sheaves.

(3) A key ingredient in the proof of the Kazhdan–Lusztig conjecture was the Kazhdan–Lusztig formula for the stalk dimensions of the cohomology of perverse sheaves. It can be written as a character formula in the Grothendieck group sense we are using here, and we describe it below.

Let $X = G/B$ denote the flag variety obtained from the simply connected semisimple complex Lie group G associated to the Lie algebra $\mathfrak{g}$ above, and consider the category $\mathcal{C}$ of perverse sheaves on X with respect to the Schubert stratification and the middle perversity **[4]**. (Thus a stratum is a Schubert cell $S(w) = BwB/B, w \in W$, and a perverse sheaf is a complex of sheaves of complex vector spaces with cohomology locally constant (thus constant) and finite-dimensional on Schubert cells, with certain support conditions satisfied.) The poset is W, with its Bruhat–Chevalley order, and the Weyl objects $V(w)$, $w \in W$, are quite easy to describe: $V(w) = i_{S(w)!}\mathbf{C}[\ell(w)]$, the extension by 0 of the constant sheaf, shifted downward as a complex in the derived category by degree $\ell(w)$ Every Weyl object $V(w)$ has a unique irreducible quotient $L(w)$, and the axioms for a highest weight category are satisfied **[28**, §5**]**. Though unnecessary in our discussion, it is a remarkable fact that $L(w)$ is the downward shift by $\ell(w)$ of the complex (extended by zero to X) defining Goreski–MacPherson intersection cohomology on the closure of $S(w)$; see **[25]** and **[33]**.

The Grothendieck group formula of Kazhdan–Lusztig **[25]** reads

$$\operatorname{ch} L(w. - 2\rho) = \sum_{y \in W} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1) \operatorname{ch} V(y. - 2\rho),$$

which is identical to the form of the Verma module Kazhdan–Lusztig conjecture above. Essentially, the latter conjecture was proved through an equivalence of categories reducing it to the above formula.

1.2 Quasihereditary algebras. Every highest weight category with finite weight poset and all objects of finite length is the category of finite-dimensional modules for a quasihereditary algebra S. Indeed, CPS introduced quasihereditary algebras for this reason, and proved that, conversely, the category of modules for a quasihereditary algebra could be viewed as a highest weight category **[31]**, **[28]**, **[7]**. We will not reproduce the axioms for a quasihereditary algebra here, but note that examples include hereditary algebras and poset algebras **[28]**, as well as all finite-dimensional algebras of global dimension two [17]. All quotient algebras of hereditary algebras are quasihereditary. Further Lie-theoretic examples of quasihereditary algebras include Schur algebras and q-Schur algebras, and their generalizations [7], [18]. CPS believes that understanding these algebras (and variations, with various degrees of added structure) will provide a good basis for understanding representations of algebraic groups in characteristic p, and finite groups of Lie type in describing or nondescribing characteristic. A new generalization of the quasihereditary notion, very relevant to the nondescribing characteristic case, is described in the last section of this paper.

Every quasihereditary algebra S has finite global dimension. The opposite algebra S^{op} is quasihereditary with the same weight poset, and the S-module $A(\lambda)$ dual to the Weyl module $V^{op}(\lambda)$ for S^{op} has the following remarkable property [7; p. 98, bottom]

$$\mathrm{Ext}^n(V(\mu), A(\lambda)) = \begin{cases} k & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have assumed, as before, that $\mathrm{End}\, L(\lambda) = k$ to simplify the statement. Using this property, and an Euler characteristic argument of Delorme, it is possible to understand why the character formulas in each of the above three cases have such a remarkably similar appearance. Moreover, by tracking in the abstract setting a version (due to MacPherson, see [33]) of the arguments used to prove the Kazhdan–Lusztig formula in the perverse sheaf case, CPS was able to provide [8] (see also [9] and [10]) the following reductions. The 'length' $\ell(\lambda)$ of a weight $\lambda = w.0$ below is the number of simple reflections in a reduced expression for w.

Theorem (The CPS reductions). *In each of the three examples above, the Lusztig conjecture or its analog (the Kazhdan–Lusztig conjecture, or Kazhdan–Lusztig formula) is equivalent to each of the following statements:*

(1) *For each* $\lambda,\ \mu \in \Lambda$, $\mathrm{Ext}^1(V(\mu), L(\lambda)) \neq 0 \Rightarrow \ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2$.

(2) For each $\lambda \in \Lambda$, and each weight λ' adjacent to λ (in the sense that the affine Weyl group or Weyl group element associated to λ' is obtained from that associated to λ by right multiplication by a simple reflection), we have $\mathrm{Ext}^1(L(\lambda'), L(\lambda)) \neq 0$. (By a duality principle, one may take here $\lambda' < \lambda$. or $\lambda' > \lambda$).

(3) For each $\lambda, \mu \in \Lambda$, the natural map

$$\mathrm{Ext}^1(L(\mu), L(\lambda)) \to \mathrm{Ext}^1(V(\mu), L(\lambda))$$

is surjective.

The most promising of these reductions is perhaps the second one, though each has its own advantages. When the Lusztig conjecture is true, versions of 1) and 3) hold with the number 1 replaced by n throughout (two replacements in 1)), and dual versions hold using $A(\mu)$, cf. **[8]**; see also **[9]**. One may study such conditions purely from the point of view of finite dimensional quasihereditary algebras, though they are far from giving us a description up to isomorphism (or a suitable weaker invariant) of the algebras involved. Until we get that far, we are somewhat in the position of trying to prove deep properties of finite simple groups without knowing what all the simple groups are.

1.3 The Lusztig program. Recently, George Lusztig **[27]** has formulated an attack on his own conjecture organized around the theory of quantum groups. Work by himself and Kazhdan relates a Lusztig-type conjecture for quantum groups at a root of unity to its validity in a category of 'negative level' representations for affine Kac–Moody Lie algebras. The latter conjecture, at least for simply laced root systems, is settled by work of Kashiwara and Tanisaki (also claimed by Casian, who acknowledges his original proof was in error) by reduction to a category of perverse sheaves on a generalized flag variety. Ignoring the non simply-laced case difficulties[1], to complete the chain, one requires a reduction from quantum groups at a pth root of unity to algebraic groups in characteristic p. This has been provided for all types by Anderson, Jantzen and Soergel **[3]** for p sufficiently large, depending on the individual root system and its rank.

Unfortunately, no specific bound whatsoever is known for p as of this writing. The problem is that p must stay away from divisors of the index in a maximal order of a certain algebra over $\mathbb{Z}$, and the algebra is constructed so indirectly that very little information on the index is available. As Jantzen himself reported at the 1994 Banff conference, this situation is simply not acceptable to finite group theory. While CPS thinks highly of the AJS work, we regard the Lusztig conjecture as open and continue to work on it.

1.4 Stratified algebras. Another aspect of the situation is that CPS wants a theory sufficiently general to be appropriate for nondescribing characteristic, in the spirit of the Dipper–James work on the q-Schur algebra. Already there has been work by Dipper and others (see **[19]**) dealing with groups other than

1 These difficulties have apparently now been handled by Kashiwara–Tanisaki.

the general linear group. While it may be that quasihereditary algebras are involved in these cases, at least in favorable characteristics, CPS conjectures a role for a slightly more general kind of algebra. This new generalization is called a *stratified* algebra **[11]**. I describe first the most basic types of stratified algebras, the algebras with a *standard* stratification, which are quite close to quasihereditary algebras, and one weaker notion, algebras (whose module category is) equipped with a *stratifying system.*

The description is quite easy to do in both cases if we just think about how we are to relax the corresponding notion (1.1) of a highest weight category: First, we relax the condition that the weights Λ form a poset, requiring only that they form a quasiposet, so that two weights λ and μ may satisfy $\lambda \leq \mu$ and $\mu \leq \lambda$ without being equal. The equivalence classes thus obtained do themselves naturally form a poset $\bar{\Lambda}$, and we let $\bar{\lambda}$ denote the element of $\bar{\Lambda}$ associated with $\lambda \in \Lambda$. Next, in condition 1) for a highest weight category, we require only that all composition factors $L(\mu)$ of the standard object $V(\lambda)$ satisfy $\mu \leq \lambda$. (So that $L(\lambda)$ may appear twice or more, along with other composition factors $L(\mu)$ with $\bar{\lambda} = \bar{\mu}$.) The second condition 2) is kept in the "standard" case, and that completes the definition for that case.

In the "stratifying system" case the inequality in 2) is relaxed to allow equality, but other relaxations are made as well: We just assume we have a system of objects $V(\lambda)$ and given projective objects $P(\lambda)$ mapping onto each of these with kernel filtered by $V(\mu)$'s with $\mu \geq \lambda$. We do not require that $V(\lambda)$ have an irreducible head, or that the quasiposet index the irreducible modules. As a replacement for the latter, we do insist that every irreducible module appear in the head of some $V(\lambda)$. Condition 1) is replaced by the requirement that there are no nonzero homomorphisms from $P(\lambda)$ to $V(\mu)$ unless $\mu \leq \lambda$.

In either case, one can prove from these conditions that the underlying algebra A has a sequence of idempotent ideals $0 = J_0 \subset J_1 \subset \cdots \subset J_n$, with $n = |\bar{\Lambda}|$, such than $\mathrm{Ext}^m_{A/J_i}(M, N) = \mathrm{Ext}^m_A(M, N)$ for all left (or right!) A/J-modules M, N, all integers $n \geq 0$, and each index i. With mention of Λ omitted, this is the *general* CPS notion of a stratified algebra, with a stratification of length n. In the "standard" case, each J_i/J_{i-1} is left projective as an A/J_{i-1}-module, and this is characteristic of standardly stratified algebras. The ideal need not be right projective, however. Unlike the quasihereditary case, there are algebras which are standardly left stratified but not standardly right stratified (even though the "general" notion of a stratified algebra is left-right symmetric). Apart from that, and the relaxed ordering inside standard modules, the theory of standardly stratified algebras is very close to that of quasihereditary algebras and highest weight categories. Note that the standard module $V(\lambda)$ is uniquely determined, in the "standard" case, as the largest quotient of $P(\lambda)$ with all composition factors $L(\mu)$ satisfying $\mu \leq \lambda$. The notion of an algebra whose (left) module category has a stratifying system, and the general notion of a stratified algebra, are both weaker, but more flexible. For instance, an algebra which has a stratification of length $n > 1$ in the "general" sense also has one of length $n - 1$, obtained by removing any of the intermediate ideals in the above chain. (There is an analog of this statement for algebras with a stratifying system.) Also, as mentioned, the "general" notion is left-right symmetric, though this is not obvious.

CPS has taken considerable trouble **[11]** to be able to recognize when an endomorphism algebra (e.g. a Schur algebra, q-Schur algebra, or future generalization) has a natural structure as a stratified algebra. This includes, of course, the quasihereditary case (easy to check in the presence of a standard stratification), but the generalizations are also interesting, and may be important. The recognition conditions are quite complicated, though simplify very considerably[2] in special cases. They involve a kind of generalized "Specht module" theory. Rather than reproduce the conditions here, I will simply give some examples from **[11]**, referring the reader to that paper for additional details, and further examples:

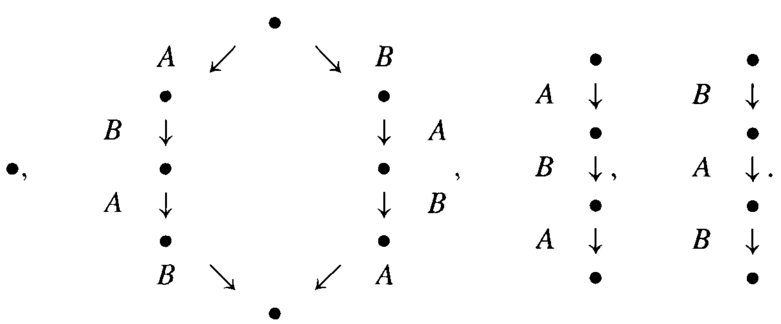
Throughout k is an algebraically closed field.

(1) *A stratification of length* 2. Let G be any nontrivial finite group. Consider the direct sum T of the trivial module and the regular permutation module over k. Then $A = \mathrm{End}_{kG}(T)$ is stratified, with $|\bar{\Lambda}| = 2$ (the *length* of the stratification). Interesting cases occur already for G the cyclic group of order 2 or the Klein four group. In the first case, A is the well-known algebra of dimension 5 which has two simple modules a and b, with projective covers $\begin{matrix} a \\ b \\ a \end{matrix}$ and $\begin{matrix} b \\ a \end{matrix}$. In the second case, A is already not quasihereditary! Its projective covers have Loewy layers as indicated by the diagrams $\begin{matrix} & a & \\ a & a & b \\ & a & \end{matrix}$ and $\begin{matrix} b \\ a \end{matrix}$.

2 The simplifications in the preprint "Stratifying endomorphism algebras over Hecke algebras", by Du, Parshall, and Scott, over $\mathbb{Z}[q, q^{-1}]$, might even be described as dramatic.

Thus, unlike the quasihereditary case, the standard object associated to a (the quotient of the first projective cover by the submodule isomorphic to the second) has a appearing with a nontrivial multiplicity. Nevertheless, inside A, the ideal generated by the idempotent associated to the second projective cover is quite nice. It is both idempotent and (left) projective. This example and the next are both standardly stratified. All standardly stratified algebras have such ideals, and their factor algebras are also standardly stratified.

(2) *A stratification of length* 3*: the dihedral group of order* 8. Let G be the dihedral group of order 8, and let T be the direct sum of the trivial module, the regular module, and the two transitive permutation modules of degree 4 associated to coset spaces of the two conjugacy classes of noncentral subgroups of order 2. We take char $k = 2$. This time the algebra $A = \mathrm{End}_{kG}\, T$ is quite difficult to visualize from the given data. The general CPS approach is to try to impose a generalized 'Specht module' filtration on T, and deduce from its properties that A is indeed stratified. Without giving full details, I will at least describe the 'Specht modules' we use. Let a and b denote generators of order two for G, and put $A = a - 1$, $B = b - 1$. Thus $A^2 = 0 = B^2$ and $ABAB = BABA$. We will diagram cyclic modules by indicating where a nontrivial action of A or B occurs, starting from a generator. (No arrow associated to a given node and label indicates a zero multiplication. Note that, if a node was reached by multiplication by A, then that node must be killed by A. A parallel statement holds for B.) Thus, the four transitive permutation modules making up T have diagrams



Here the single node • is also representative of the unique irreducible (trivial) module for the group G, and the above diagrams give refined Loewy series pictures. In the CPS set-up, each of these indecomposable components of T has a 'Specht filtration', which turns into the required filtration of projective covers by standard modules for the algebra $A = \mathrm{End}_{kG}(T)$ under the contravariant functor $\mathrm{Hom}_{kG}(-, T)$. (This functor is not exact, but a filtration of T still induces a filtration of A.) Moreover, each component above has a distinguished 'Specht' submodule, though it is possible for two such 'Specht' modules to be isomorphic. For the first and second components above, the Specht submodule is the 1-dimensional trivial module, while for the third and fourth component, the Specht submodule is the unique 3-dimensional submodule. The reader will observe that, when the bottom and top trivial modules are eliminated from the second component, the remainder is the direct sum of the Specht modules associated to the third and fourth component. This puts all 'Specht filtrations' in evidence. The reader may consult **[11]** for some general machinery to check that these filtrations are transformed into the required standard module filtrations for the indecomposable projective components of A (or can attempt a direct verification starting from the filtrations and functor we have given).

CPS believes something general is happening here for all Coxeter groups, that a similar construction always leads to a nontrivial stratified algebra. We have conjectured the following:

Conjecture. *Let W be a finite Coxeter group with distinguished generating set S, $|S| > 1$. For $J \subseteq S$, let T_J denote the permutation module for kW on the cosets $\{W_J w\}_{w\in W}$. Put $T = \bigoplus_{J\subseteq S} T_J$ and $A = \mathrm{End}_{kW} T$. Then A is stratified with respect to a quasiposet Λ with $|\bar{\Lambda}| \geq 3$.*

For a more detailed statement of the conjecture, see **[11]**. We believe the stratification arises from a stratifying system, and that $\bar{\Lambda}$ may be assumed to have a largest and smallest element, containing only one element each as equivalence classes in Λ, with these elements associated to T_S (the trivial module) and the sign module submodule of T_ϕ. The stronger form of the conjecture also has as a consequence the existence of certain known resolutions, one of which is the Coxeter complex. As the final version of this paper is being readied for press, it appears that Du, Parshall and Scott will soon prove the stronger conjecture, and a q-analog.[3]

CPS also expects (with Jie Du) that an enlarged version of A will have a standard stratification related to the filtration of the T_J's by dual left cell modules, in the sense of Kazhdan–Lusztig, of length equal to the number of two-sided cells. The algebra A itself exhibits such a stratification in the order 8 dihedral case above. This is a special case of a Weyl group of type B. In

3 This has now been done, in the preprint described in the previous footnote.

this case Jie Du and I are working on another natural enlargement that may be quasihereditary.[4]

As mentioned above, it also seems likely that a q-analog of the conjecture holds for Hecke algebras, a possibility which makes the conjecture and stratified algebras quite relevant to nondescribing characteristic theory. This already important area of research will become even more central once the problems posed by the Lusztig conjecture itself are solved.

References

[1] J. Alperin, Cohomology in representation theory, Proc. Symp. Pure Math. 47 (1987). 3–11.

[2] M. Aschbacher and L. Scott, Maximal subgroups of finite groups, J. Algebra 92 (1985), 44–80.

[3] H. Andersen, J. Jantzen, and W. Soergel, Representations of quantum groups at a p th root of unity and of semisimple algebraic groups in characteristic p: Independence of p, Astérisque 220 (1994).

[4] A. Beilinson, J. Bernstein, and P. Deligne, Analyse et topologie sur les espaces singulares, Astérisque 100 (1982).

[5] A. Beilinson and J. Bernstein, Localisation des $\mathcal{G}$-modules, C. R. Acad. Sci. Paris 292 (1981), 15–18.

[6] J. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, Invent. Math. 64 (1981), 387–410.

[7] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. 391 (1988), 85–99.

[8] E. Cline, B. Parshall and L. Scott, Abstract Kazhdan–Lusztig theories, Tôhoku Math. 45 (1993), 511–534.

[9] E. Cline, B. Parshall and L. Scott, Infinitesimal Kazhdan–Lusztig theories, Contemp. Math. 39 (1992), 43–73.

[10] E. Cline, B. Parshall and L. Scott, Simulating perverse sheaves in modular representation theory, Proc. Symp. Pure Math. 56 (1994), 63–104.

[11] E. Cline, B. Parshall and L. Scott, Stratifying endomorphism algebras, Mem. Amer. Math. Soc. 591, Vol. 124, 1996.

[12] C. Curtis, Irreducible representations of finite groups of Lie type, J. Reine Angew. Math. 219 (1965), 180–199.

[13] C. Curtis, Modular representations of finite groups with a split (B,N) pair, Seminar on algebraic groups and related finite groups (A. Borel. ed.), Lecture Notes in Math. 131, Springer-Verlag, 1970, 57–95.

4 This is true. The preprint by Du and Scott is entitled "The q-Schur^2 algebra".

[14] R. Dipper, Polynomial representations of finite general linear groups in non-describing characteristic, Prog. in Math. 95 (1991), 343–370.

[15] R. Dipper and G. D.James, The q-Schur algebra, J. London Math. Soc. 59 (1989), 23–50.

[16] R. Dipper and G. D. James, q-tensor space and q-Weyl modules, Trans. Amer. Math. Soc. 327 (1991), 251–282.

[17] V. Dlab and C. Ringel, Auslander algebras are quasi-hereditary, J. London Math. Soc. 39 (1989), 457–466.

[18] J. Du and L. Scott, Lusztig conjectures, old and new, I, J. Reine Angew. Math. 455 (1994), 141–182.

[19] R. Geck and G. Hiss, Modular representations of finite groups of Lie type in non-describing characteristic, in: Finite reductive groups: Related structures and representations (M. Cabanes, ed.), Birkhäuser, 1997, 195–249.

[20] J. Humphreys, Introduction to Lie algebras and representation theory, Grad. Texts in Math. 9, Springer-Verlag, 1970.

[21] G. D. James, The irreducible representations of the finite general linear groups, Proc. London Math. Soc. 60 (1990), 225–265.

[22] J. Jantzen, Representations of algebraic groups, Academic Press, 1987.

[23] S. Kato, On the Kazhdan–Lusztig polynomials for affine Weyl groups, Adv. in Math. 55 (1985), 103–130.

[24] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math.83 (1979), 165–184.

[25] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, Proc. Symp. Pure Math. 36 (1980), 185–203.

[26] G. Lusztig, Some problems in the representation theory of finite Chevalley groups, Proc. Symp. Pure Math.37 (1980), 313–317.

[27] G. Lusztig, Modular representations and quantum groups, Contemp. Math.82 (1989), 59–77.

[28] B. Parshall and L. Scott, Derived categories, quasi-hereditary algebras, and algebraic groups, Mathematical Lecture Notes Series 3, Carleton University, 1988, 1–105.

[29] F. Richen, Modular representations of split BN pairs, Trans. Amer. Math. Soc. 140 (1969), 435–460.

[30] L. Scott, Representations in characteristic p, Proc. Symp. Pure Math. 37 (1980), 319–331.

[31] L. Scott, Simulating algebraic geometry with algebra, I: Derived categories and Morita theory, Proc. Symp. Pure Math. 152 (1987), 271–281.

[32] L. Scott, Quasihereditary algebras and Kazhdan–Lusztig theory, Finite dimensional algebras and related topics (V. Dlab and L. Scott, eds.), Kluwer, 1994, 293–308.

[33] T. Springer, Quelques applications de la cohomologie d'intersection, Bourbaki Sem. (1981/1982).

[34] R. Steinberg, Representations of algebraic groups, Nagoya Math. J. 22 (1963), 33–56.

[35] R. Steinberg, Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80, Amer. Math. Soc., Providence, R.I., 1968.

Department of Mathematics
Unversity of Virginia
Charlottesville, VA 22903
U.S.A.
Email: lls2l@virginia.edu