Transformation Groups, Vol. 9, No. 3, 2004, pp. 213–236

ON EXT-TRANSFER FOR ALGEBRAIC GROUPS

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We dedicate this paper to the memory of our friend, Gordon Keller

Abstract. This paper builds upon the work of Cline [C] and Donkin [D1] to describe explicit equivalences between some categories associated to the category of rational modules for a reductive group G and categories associated to the category of rational modules for a Levi subgroup H. As an application, we establish an Ext-transfer result from rational G-modules to rational H-modules. In case $G = \mathrm{GL}_n$, these results can be illustrated in terms of classical Schur algebras. In that case, we establish another category equivalence, this time between the module category for a Schur algebra and the module category for a union of blocks for a natural quotient of a larger Schur algebra. This category equivalence provides a further Ext-transfer theorem from the original Schur algebra to the larger Schur algebra. This result, announced in [PS2, (6.4b)], extends to the category level the decomposition number method of Erdmann [E2]. Finally, we indicate (largely without proof) some natural variations to situations involving quantum groups and q-Schur algebras.

1. Introduction

Let G be a reductive algebraic group over an algebraically closed field k. Let H be a Levi subgroup of G. In independent work, Cline [C] and Donkin [D1] present different ways to relate the representation theory of H to that of G. Actually, the approach in [C] works with infinitesimal thickenings G_rT and H_rT ($r \geqslant 1$) (defined by a Frobenius map $F : G \to G$) and uses Harish-Chandra induction to pass up from H_rT to G_rT. The approach in [D1] proceeds in the opposite direction, coming down to H from G by a weight truncation operator (first studied by Jantzen [Jan1], [Jan2]). In both approaches,

*Supported by NSF.
**Supported by NSF.
Received September 26, 2003. Accepted February 15, 2004.

a fixed coset $\Omega := \nu + \mathbb{Z}\Phi_H$ of the root lattice of H in the weight lattice X of G plays a key role. In [C], there is a full embedding from the derived category of rational H_rT-modules whose composition factors have high weights in Ω into the derived category of rational G_rT-modules. Thus, under this embedding $\text{Ext}^\bullet$ groups are preserved. The theory in [D1] is more character-theoretic, using the natural truncation operator π_Ω on weights of G-modules, throwing away weights not in Ω: this operation preserves irreducibility, standard modules, and decomposition numbers.

The main goal of this paper is to establish that the above truncation procedure also has some remarkable homological properties. These properties, in turn, derive from an equivalence between a certain category of rational G-modules and a category of rational H-modules. The homological properties, of considerable interest in their own right, also contain Donkin's result on decomposition numbers, which may now be seen in terms of an Euler characteristic property.

Our paper is organized as follows. Section 2 collects together some preliminary material concerning general highest weight categories. In Section 3, we prove the main results just mentioned. Section 4 shows that a special case of the procedure amounts to passage from homogeneous polynomial modules of degree r for $\text{GL}_r(k)$ to homogeneous modules of degree r for $\text{GL}_n(k)$, i.e., passage from $S(r,r)$-mod to $S(n,r)$-mod, when $n \leqslant r$. Here $S(n,r)$ denotes the Schur algebra of bidegree (n,r) over the field k. In this way, the results of Section 3 cast the classical Schur algebra situation in a broader context. Some of the decomposition number applications in [D1] foreshadow this context, though our interests here are more homological.

Section 5 presents a different kind of Ext-transfer result that arises in the case of Schur algebras, i.e., when $G \cong \text{GL}_n(k)$. This transfer, which was discussed in [PS2, §6] in special cases, actually arises from an interesting (and new) Morita equivalence between the Schur algebra $S(r,r)$ and an algebra which is a product of various blocks of a natural quotient algebra of $S(s,s)$, where $s = pr + (p-1)r(r-1)/2$; see Theorem 17. Then Corollary 18 produces, by means of a standard recollement argument, a further Ext-transfer. As suggested already for the more special results in [PS2], this isomorphism of Ext-groups provides a natural cohomological analogue of Erdmann's decomposition number method [E2]. The present paper even extends this to the categorical level.

Finally, we briefly indicate in Section 6 how some of these ideas extend to the context of quantum groups and q-Schur algebras.

We thank both referees of this paper for providing detailed comments which the authors found useful.

2. Preliminaries

1. Quasi-hereditary algebras

We assume familiarity with the basic theory of quasi-hereditary algebras and highest weight categories [CPS1]. Actually, it is convenient and simplifying in our context to consider only finite-dimensional modules. Thus, "highest weight category" means the category of finite length objects in a highest weight category in the sense of [CPS1]. This convention will be convenient, for example, when dealing with the category of rational modules for a reductive algebraic group.

Let A be a quasi-hereditary algebra over an algebraically closed field k. Thus, the category of finite-dimensional A-modules, denoted A-mod, is a highest weight category with respect to a poset $(\Lambda, \leqslant)$ (i.e., the *weight poset*). We remark that in this case, A-mod is a highest weight category in the full sense of [CPS1].

The set Λ indexes the set of isomorphism classes of irreducible objects in A-mod. For $\lambda \in \Lambda$, let $L(\lambda) = L_A(\lambda)$ be an irreducible A-module representing the class indexed by λ and $\Delta(\lambda) = \Delta_A(\lambda)$ (resp., $\nabla(\lambda) = \nabla_A(\lambda)$) denote the corresponding standard (resp., costandard) objects in A-mod. Thus $\Delta(\lambda)$ has head $L(\lambda)$ and $\nabla(\lambda)$ has socle $L(\lambda)$. We will often work with modules M which have a Δ-filtration, i.e., there exists a finite filtration $0 = F^0 \subset F^1 \subset \cdots \subset F^n = M$ of M by submodules such that each section F^i/F^{i-1} is isomorphic to some standard module $\Delta(\lambda_i)$. Similarly, we can speak of modules having a ∇-filtration.

Let $\mathcal{C}$ be an arbitrary highest weight category with weight poset Λ. Recall that two indecomposable objects M, M' in $\mathcal{C}$ belong to the same *block* provided there is a finite sequence $M = M_1, M_2, \dots, M_n = M'$ of indecomposable objects in $\mathcal{C}$ so that each adjacent pair M_i, M_{i+1} shares a common composition factor. At the level of the weight poset, a subset $B \subseteq \Lambda$ is called a *block* provided the set of irreducible objects $L(\xi)$ with $\xi \in B$ comprise the set of irreducible modules in a block of $\mathcal{C}$.

The following result will be useful; see [CPS3, Lemma 2.2].

Lemma 1. *Let A be a quasi-hereditary algebra. For $\lambda, \mu \in \Lambda$,*

$$\dim \operatorname{Ext}^i_A(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}\delta_{i,0}.$$

In establishing our main results, we will require the Comparison Theorem [PS1, Theorem 5.8] stated below. Let A_1, A_2 be quasi-hereditary algebras over an algebraically closed field k. Suppose that sets of isomorphism classes of irreducible objects in A_1-mod and A_2-mod are both indexed by the same set Λ, and that $\leqslant$ is a poset structure on Λ making both A_1-mod and A_2-mod into highest weight categories. For $i = 1, 2$, and $\lambda \in \Lambda$, let $L_i(\lambda) = L_{A_i}(\lambda), \Delta_i(\lambda) = \Delta_{A_i}(\lambda)$ and $\nabla_i(\lambda) = \nabla_{A_i}(\lambda)$, respectively.

Theorem 2. *Let $E : A_1\text{-mod} \to A_2\text{-mod}$ be an exact, additive functor. Suppose that for all $\lambda \in \Lambda$,*

$$E\nabla_1(\lambda) \cong \nabla_2(\lambda) \quad \text{and} \quad E\Delta_1(\lambda) \cong \Delta_2(\lambda).$$

Then E is an equivalence of categories. In particular, the algebras A_1 and A_2 are Morita equivalent.

2. Some homological algebra

Let A be a quasi-hereditary algebra over an algebraically closed field k. Let Γ be an ideal in the weight poset Λ (i.e., $\mu \leqslant \lambda \in \Gamma \Rightarrow \mu \in \Gamma$), and denote the coideal $\Lambda \setminus \Gamma$ by Ω. Let A-mod$[\Gamma]$ be the full subcategory of A-mod having as objects those A-modules M all of whose composition factors are isomorphic to various $L(\lambda)$, $\lambda \in \Gamma$. Let $i_* : A\text{-mod}[\Gamma] \to A\text{-mod}$ be the natural exact full embedding. Then A-mod$[\Gamma]$ is a highest weight category with poset Γ; in addition, $A\text{-mod}[\Gamma] \cong A/J\text{-mod}$ for some idempotent ideal $J \trianglelefteq A$. Also, A-mod$[\Gamma]$ is a Serre subcategory of A-mod; the quotient category of A-mod by A-mod$[\Gamma]$ is denoted by A-mod(Ω) because it is a highest weight category with weight poset Ω. Let $j^* : A\text{-mod} \to A\text{-mod}(\Omega)$ be the canonical exact quotient functor. If $J = AeA$ for an idempotent $e \in J$, then $A\text{-mod}(\Omega) \cong eAe\text{-mod}$.

Under this identification, the functor j^* identifies with the functor A-mod $\to eAe$-mod which sends an A-module M to the eAe-module eM. (See [P, §§1,2] for further discussion.) In addition, $j^*M = 0$ if and only if M lies in the image of the functor i_*.

For $\gamma \in \Gamma$ (resp., $\omega \in \Omega$), we write $L_\Gamma(\gamma)$, $\Delta_\Gamma(\gamma)$, and $\nabla_\Gamma(\gamma)$ (resp., $L_\Omega(\omega)$, $\Delta_\Omega(\omega)$ and $\nabla_\Omega(\omega)$) for the corresponding irreducible, standard and co-standard objects in A/J-mod $\cong A$-mod$[\Gamma]$ (resp., A-mod$(\Omega) \cong eAe-$mod). For $\gamma \in \Gamma$, we have $i_*L_\Gamma(\gamma) \cong L(\gamma)$, $i_*\Delta_\Gamma(\gamma) \cong \Delta(\gamma)$ and $i_*\nabla_\Gamma(\gamma) \cong \nabla(\gamma)$. Also, for $\omega \in \Omega$, $j^*L(\omega) \cong L_\Omega(\omega)$, $j^*\Delta(\omega) \cong \Delta_\Omega(\omega)$ and $j^*\nabla(\omega) \cong \nabla_\Omega(\omega)$.

This set-up leads to a "recollement diagram":

$$D^b(A\text{-mod}[\Gamma]) \;\overset{\overset{\mathbf{i}^*}{\longleftarrow}}{\underset{\underset{\mathbf{i}^!}{\longleftarrow}}{\xrightarrow{\ \mathbf{i}_*\ }}}\; D^b(A\text{-mod}) \;\overset{\overset{\mathbf{j}_!}{\longleftarrow}}{\underset{\underset{\mathbf{j}_*}{\longleftarrow}}{\xrightarrow{\ \mathbf{j}^*\ }}}\; D^b(A\text{-mod}(\Omega)) \tag{1}$$

for the bounded derived categories of the respective highest weight categories. Here $\mathbf{i}_*$ and $\mathbf{j}^*$ denote the functors on derived categories induced by the exact functors i_* and j^* on the module categories. In particular, if $X^\bullet$ belongs to $D^b(A/J\text{-mod}) \cong D^b(A\text{-mod}[\Gamma])$, the complex $\mathbf{i}_*X^\bullet$ is obtained by applying i_* to each term of the complex $X^\bullet$. Similarly, given a complex $Y^\bullet \in D^b(A\text{-mod})$, $\mathbf{j}^*Y^\bullet \in D^b(eAe\text{-mod}) \cong D^b(A\text{-mod}(\Omega))$ is represented by the complex $eY^\bullet$ obtained by applying the functor j^* to each term of $Y^\bullet$. The functor $\mathbf{i}_*$ is a full embedding of derived categories; thus,[1]

$$\mathrm{Ext}^\bullet_{A/J}(M,N) \cong \mathrm{Ext}^\bullet_A(M,N), \quad M,N \in A/J\text{-mod}. \tag{2}$$

The other functors in (1) are defined as adjoints of $\mathbf{i}_*$ and $\mathbf{j}^*$: $(\mathbf{i}^*, \mathbf{i}_*, \mathbf{i}^!)$ and $(\mathbf{j}_!, \mathbf{j}^*, \mathbf{j}_*)$ form adjoint triples, with $\mathbf{j}_!, \mathbf{j}_*$ full embeddings of derived categories. Thus, $\mathbf{i}^*$ (resp., $\mathbf{i}^!$) is left (resp., right) adjoint to $\mathbf{i}_*$. Similarly, $\mathbf{j}_!$ (resp., $\mathbf{j}_*$) is the left (resp., right) adjoint to $\mathbf{j}^*$. More specifically, $\mathbf{i}^*$ (resp., $\mathbf{i}^!$) is realized as the left (resp., right) derived functor of the module functor $i^* : A\text{-mod} \to A/J\text{-mod}$ (resp., $i^! : A\text{-mod} \to A/J\text{-mod}$) which associates to any A-module M the largest quotient module (resp., submodule) which lies in the full subcategory A/J-mod of A-mod. Similarly, $\mathbf{j}_* := \mathbb{R}\mathrm{Hom}_{eAe}(eA, -)$ is the right derived functor of $j_* = \mathrm{Hom}_{eAe}(eA, -) : eAe\text{-mod} \to A\text{-mod}$, while $\mathbf{j}_! := Ae \otimes^{\mathbb{L}}_{eAe} -$ is the left derived functor of $j_! := Ae \otimes_{eAe} - : eAe\text{-mod} \to A\text{-mod}$. For later use, observe that $j_!$ (resp., j_*) is right (resp., left) exact. See [CPS1] and [P] for more details.

The following result shows that the standard (resp., costandard) modules for the highest weight category $A-$mod(Ω) behave well with respect to the adjoint $\mathbf{j}_!$ (resp., $\mathbf{j}_*$). This result will be used in §3 below, but also has independent interest.

Lemma 3. *For any $\omega \in \Omega$,*

$$\mathbf{j}_!\Delta_\Omega(\omega) \cong j_!\Delta_\Omega(\omega) \cong \Delta(\omega), \quad \mathbf{j}_*\nabla_\Omega(\omega) \cong j_*\nabla_\Omega(\omega) \cong \nabla(\omega).$$

Proof. For an eAe-module M,

$$\mathbf{j}_!M \cong Ae \otimes^{\mathbb{L}}_{eAe} M \in D^b(A\text{-mod}).$$

[1]In general, $\mathrm{Ext}^i_A(M,N) \cong \mathrm{Hom}_{D^b(A\text{-mod})}(M, N[i])$ for A-modules M,N. Here and elsewhere we regard A-mod as a subcategory of $D^b(A\text{-mod})$ consisting of complexes concentrated in degree 0, while $N[i]$ is the complex which is 0 in all degrees except for N in degree $-i$.

Thus, if M is projective, $\mathbf{j}_!M \cong j_!M := Ae\otimes_{eAe}M \in A$-mod is projective. In particular, if $\omega \in \Omega$ is maximal, then $\Delta_\Omega(\omega)$ is projective, so that $\mathbf{j}_!\Delta_\Omega(\omega) \cong j_!\Delta_\Omega(\omega) \in A$-mod is projective. Because $j^*L(\lambda) \cong L_\Omega(\lambda)$ provided $\lambda \in \Omega$ and equals 0 otherwise, we have

$$\mathrm{Hom}_A(j_!\Delta_\Omega(\omega), L(\lambda)) \cong \mathrm{Hom}_{eAe}(\Delta_\Omega(\omega), j^*L(\lambda)) \neq 0$$

for any $\lambda \in \Lambda$ if and only if $\lambda = \omega$ (in which case, this Hom-space is one-dimensional). It follows that $j_!\Delta_\Omega(\omega)$ is the projective indecomposable A-module corresponding to ω. Since ω is also maximal in Λ, this gives that $j_!\Delta_\Omega(\omega) \cong \Delta(\omega)$, as required.

The opposite algebra A^{op} is also quasi-hereditary [CPS1, p. 93]—in fact, the axioms for a quasi-hereditary algebra are left–right symmetric. Using linear duality $M \mapsto M^*$ to identify A^{op}-mod with the category of right A-modules, we see that A^{op}-mod is a highest weight category with weight poset Λ, standard objects $\nabla(\lambda)^*$ and costandard objects $\Delta(\lambda)^*$. Let $M \in eAe$-mod. Since

$$\mathrm{Hom}_k(Ae \otimes_{eAe} M, k) \cong \mathrm{Hom}_{eAe}(Ae, \mathrm{Hom}_k(M, k)),$$

it follows from Lemma 1 and the fact that Ae has a Δ_Ω-filtration in the highest weight category mod-eAe of right eAe-modules that,

$$\mathbf{j}_!M \cong Ae \otimes^{\mathbb{L}}_{eAe} M \cong Ae \otimes_{eAe} M := j_!M,$$

for any $M \in A$-mod(Ω) which has a Δ_Ω-filtration. In particular, $\mathrm{Tor}_1^{eAe}(Ae, M) = 0$.

The previous paragraph shows that $\mathbf{j}_!\Delta_\Omega(\omega) \cong j_!\Delta_\Omega(\omega)$ for all $\omega \in \Omega$. So we must show that $j_!\Delta_\Omega(\omega) \cong \Delta(\omega)$ for all $\omega \in \Omega$. We have already shown that this holds for $\omega \in \Omega$ maximal, so we proceed by induction. So, suppose that $\omega \in \Omega$ is such that $j_!\Delta_\Omega(\omega') \cong \Delta(\omega')$ for any $\omega' \in \Omega$ satisfying $\omega' > \omega$. Let $P_\Omega(\omega)$ be the projective cover in A-mod(Ω) of $\Delta_\Omega(\omega)$ and form the exact sequence $0 \to C(\omega) \to P_\Omega(\omega) \to \Delta_\Omega(\omega) \to 0$, so that $C(\omega)$ has a Δ_Ω-filtration in A-mod(Ω) with sections of the form $\Delta_\Omega(\omega')$, $\omega' > \omega$. Applying the right exact module functor $j_!$ and using the previous paragraph, we obtain an exact sequence $0 \to j_!C(\omega) \to j_!P_\Omega(\omega) \to j_!\Delta_\Omega(\omega) \to 0$ in A-mod. By our assumption on ω, $j_!C(\omega)$ has a Δ-filtration with sections of the form $\Delta(\omega')$, $\omega' > \omega$. Also, observe that $j_!P_\Omega(\omega)$ is isomorphic to the projective cover $P(\omega)$ of $L(\omega)$ (or of $\Delta(\omega)$) in the category A-mod. In fact, since $j_!$ has an exact right adjoint j^*, it follows that $j_!P_\Omega(\omega)$ is projective in A-mod. Also, if $\xi \in \Omega$, $\dim \mathrm{Hom}_A(j_!P_\Omega(\omega), L(\xi)) = \dim \mathrm{Hom}_{eAe}(P_\Omega(\omega), j^*L(\xi))$ equals 1 if $\xi = \omega$ (since $j^*L(\omega) \cong L_\Omega(\omega)$ in that case) and is otherwise equal to 0. Hence, $j_!P_\Omega(\omega) \cong P(\omega)$. Since $j^*P(\omega) \cong P_\Omega(\omega)$ [P, Lemma 1.2(c)], the isomorphism $j_!P_\Omega(\omega) \xrightarrow{\sim} P(\omega)$ is given by the adjunction morphism $j_!j^*P(\omega) \to P(\omega)$.

Let $Q(\omega)$ be the kernel of the natural surjection $P(\omega) \twoheadrightarrow \Delta(\omega)$, and form the commutative diagram in A-mod

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_!C(\omega) & \longrightarrow & j_!P_\Omega(\omega) & \longrightarrow & j_!\Delta_\Omega(\omega) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\pi} & & \\
0 & \longrightarrow & Q(\omega) & \longrightarrow & P(\omega) & \longrightarrow & \Delta(\omega) & \longrightarrow & 0
\end{array}$$

with exact rows. In this diagram, π is the adjunction morphism $j_!j^*\Delta(\omega) \cong j_!\Delta_\Omega(\omega) \to \Delta(\omega)$, while the morphism σ is induced by π. By the snake lemma, the kernel K of the surjective map π is isomorphic to the cokernel of the injective map σ. Because $\Delta_\Omega(\omega) \cong j^*\Delta(\omega)$, $j^*K = 0$. Thus, the A-module K has the form $K = i_*K'$ for some $K' \in A/J$-mod. On the other hand, $i^*Q(\omega) = 0$, using the right exactness of i^* and the facts that $Q(\omega)$ has a Δ-filtration by $\Delta(\tau)$, $\tau \in \Omega$, and $i^*\Delta(\tau) = 0$ for any such τ. (Note that $i^*M \cong M/JM$ for any M.) Thus, $\mathrm{Hom}_A(Q(\omega), K) = \mathrm{Hom}_A(Q(\omega), i_*K') \cong \mathrm{Hom}_{A/J}(i^*Q(\omega), K') = 0$. It follows that $K = 0$, showing that $j_!\Delta_\Omega(\omega) \cong \Delta(\omega)$, as required.

This completes the proof that $\mathbf{j}_!\Delta_\Omega(\omega) \cong j_!\Delta_\Omega(\omega) \cong \Delta(\omega)$. The dual statement for $\nabla(\omega)$ is obtained in a dual fashion. □

3. Tilting modules and the Ringel dual

We merely summarize a few key facts. For more details, see [R], [D3], [CPS5, §4].

Let A-mod be a highest weight category with weight poset $(\Lambda, \leqslant)$. For $\lambda \in \Lambda$, let $Y(\lambda)$ be the indecomposable tilting module of highest weight λ. Thus, $Y(\lambda)$ has two filtrations: $0 = F^0 \subset F^1 \subset \cdots \subset F^m = Y(\lambda)$ and $Y(\lambda) = G_0 \supset G_1 \supset \cdots \supset G_n = 0$ in which $F^i/F^{i-1} \cong \Delta(\lambda_i)$, $0 < i \leqslant m$, (resp., $G_{j-1}/G_j \cong \nabla(\mu_j)$, $0 < j \leqslant n$) for some $\lambda_i, \mu_j \in (-\infty, \lambda] = \{\mu \in \Lambda \mid \mu \leqslant \lambda\}$ with $\lambda_1 = \lambda = \mu_1$. Necessarily, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) appears with multiplicity 1 as a section in the filtration $F^\bullet$ (resp., $G_\bullet$).

Fix a *complete* tilting module

$$Y = \bigoplus\nolimits_{\lambda\in\Lambda} Y(\lambda)^{\oplus r_\lambda}, \tag{3}$$

which just means that each multiplicity r_λ is positive. The *Ringel dual* of A-mod is the category E-mod, where

$$E = \mathrm{End}_A(Y). \tag{4}$$

Although the algebra E depends on the choice of the positive integers r_λ, a different choice leads to a Morita equivalent algebra; it is in this sense that we speak of *the* Ringel dual of A-mod. The category E-mod is itself a highest weight category with weight poset $(\Lambda, \leqslant^{\mathrm{op}})$, the opposite of the poset for A-mod. The standard modules $\Delta_E(\lambda)$ for E-mod can be described as follows in terms of the standard modules $\Delta(\lambda)$ for A-mod. Regard Y as a left E-module, and form the contravariant "tilting functor"

$$T_A = \mathrm{Hom}_A(-, Y) : A\text{-mod} \longrightarrow E\text{-mod}. \tag{5}$$

Then

$$\Delta_E(\lambda) \cong T_A(\Delta(\lambda)) := \mathrm{Hom}_A(\Delta(\lambda), Y). \tag{6}$$

In addition, Y is a full tilting module in E-mod and the double centralizer property $\mathrm{End}_E(T) \cong A$ holds. In particular, A-mod identifies with the Ringel dual of the highest weight category E-mod.

Lemma 4. (a) *Suppose $M \in A$-mod has a Δ-filtration* $0 = F^0 \subset F^1 \subset \cdots \subset F^m = M$ *with sections $F^i/F^{i-1} \cong \Delta(\lambda_i)$, $\lambda_i \in \Lambda$. Then*

$$0 = T_A(M/F^m) \subset T_A(M/F^{m-1}) \subset \cdots \subset T_A(M/F^0) = T_AM$$

forms a Δ-filtration of T_AM with sections

$$T_A(M/F^{i-1})/T_A(M/F^i) \cong \Delta_E(\lambda_i).$$

(b) *For $\lambda \in \Lambda$, let $P(\lambda)$ be the projective indecomposable A-module of highest weight λ. Then we have an isomorphism $T_A P(\lambda) \cong Y_E(\lambda)$, the indecomposable tilting module in E-mod of highest weight λ.*

(c) *For $\lambda \in \Lambda$, $T_A Y(\lambda) \cong P_E(\lambda)$, the projective indecomposable E-module of highest weight λ.*

(d) *If Proj_A denotes the full subcategory of A-mod whose objects are projective modules, then $T_E \circ T_A|_{\mathrm{Proj}_A} \cong \mathrm{Id}_{\mathrm{Proj}_A}$, the identity functor on Proj_A. Similarly, if Tilt_A denotes the full subcategory of A-mod whose objects are tilting modules, then $T_E \circ T_A|_{\mathrm{Tilt}_A} \cong \mathrm{Id}_{\mathrm{Tilt}_A}$.*

Proof. Part (a) follows directly from Lemma 1 and (6). It is easy to see that $T_A P(\lambda)$ is an indecomposable summand of Y as an E-module. Hence, $T_A P(\lambda)$ is an indecomposable tilting module in E-mod. But $P(\lambda)$ has a Δ-filtration in which the sections are all of the form $\Delta(\mu)$, $\mu \geqslant \lambda$. In addition, $\Delta(\lambda)$ occurs exactly one time, namely, as the "top" section. Thus, by (a) and (6), $T_A P(\lambda)$ has a Δ_E-filtration in which $\Delta_E(\lambda)$ is a section and λ is maximal among all μ in the opposite poset $(\Lambda, \leqslant^{\mathrm{op}})$ for which $\Delta_E(\mu)$ is a section of $T_A P(\lambda)$. Thus, $T_A P(\lambda) \cong Y_E(\lambda)$, proving (b). The proof of (c) is similar; we leave the details to the reader. Part (d) also follows. □

We remark that the above result, in a slightly different form and in terms of basic algebras, already appears in [R, Theorem 6 and Lemma 7]. The difference is that [R] works with the covariant functor $\mathrm{Hom}_A(Y, -)$, while we have found it more convenient to cast things in terms of our contravariant functor (5).

Example 1. We assume the reader is familiar with the theory of q-Schur algebras. The special case of $q = 1$ (classical Schur algebras) will be discussed in §4 and the general case briefly touched upon in §5. Below we will freely use some of the standard notation from §§4,5. We briefly indicate how the Ringel dual works in this setting. So let $S_q(n, r)$ be the q-Schur algebra of bidegree (n, r) over an algebraically closed field k. Here $0 \neq q \in k$. When $q = 1$, we denote $S_q(n, r)$ just by $S(n, r)$; see §4. Also, $S_q(n, r)$-mod is a highest weight category with weight poset $\Lambda^+(n, r)$, the set of partitions of r of length at most n, given its dominance partial order $\trianglelefteq$. We have the following explicit description of the Ringel dual of $S_q(n, r)$-mod. This result is proved independently in [D4] and [DPS] (and in [D3] when $q = 1$, which will be the case of most interest in this paper).

Theorem 5. *The Ringel dual* $E_q(n, r)$-mod *of* $S_q(n, r)$-mod *is a highest weight category with poset $\Lambda^+(n, r)'$, the ideal in $\Lambda^+(r) = \Lambda^+(r, r)$ consisting of all dual partitions λ' for $\lambda \in \Lambda^+(n, r)$.*

Furthermore, we have the following explicit identifications of the Ringel dual:

(a) *If $n \geqslant r$, there is an equivalence of categories*

$$G : E_q(n, r)\text{-mod} \xrightarrow{\sim} S_q(n, r)\text{-mod}, \quad \Delta_{E_q(n,r)}(\lambda) \mapsto \Delta(\lambda').$$

(b) *If $n < r$, there is an equivalence of categories*

$$G : E_q(n, r)\text{-mod} \xrightarrow{\sim} S_q(r, r)\text{-mod}[\Lambda^+(n, r)'], \quad \Delta_{E_q(n,r)}(\lambda) \mapsto \Delta(\lambda').$$

3. The main results

In this section, we work in the setting of reductive groups over an algebraically closed field k. Thus, our point of view here is closest to that in [D1].

Fix a connected, reductive algebraic group G over k with maximal torus T and Borel subgroup $B \supseteq T$; let $H \supseteq T$ be a Levi subgroup of a parabolic subgroup $P \supseteq B$. For simplicity, we will assume that the derived group of G is simply connected; hence, the derived group of H is also simply connected. We let $X = \mathrm{Hom}(T, \mathbb{G}_m)$ denote the weight lattice of T and $X^\vee = \mathrm{Hom}(\mathbb{G}_m, T)$ the dual lattice of coweights. The pairing $\langle\ ,\ \rangle : X \times X^\vee \to \mathbb{Z} \cong \mathrm{End}(\mathbb{G}_m)$ is the usual pairing induced by composition.

The root groups of G relative to T are the minimal elements in the set of non-identity unipotent subgroups of G normalized by T. Each root group U_α is isomorphic to $\mathbb{G}_a$; the conjugation action of T on U_α defines a unique root $\alpha \in X$. Let Φ denote the root system of G, i.e., the set of roots relative to T. The root system Φ_H of H consists of those roots $\beta \in \Phi$ such that $U_\beta \subset H$. See [Sp] for more discussion.

The Borel subgroup B determines a unique positive system of roots, $\Phi^+ = \{\alpha \in \Phi \mid U_\alpha \subset B\}$, together with its associated system of simple roots $\Pi \subset \Phi^+$. Moreover, $\Phi^+_H = \Phi^+ \cap \Phi_H$ and $\Pi_H = \Pi \cap \Phi_H$ are the positive and simple systems of roots in Φ_H determined by the Borel subgroup $B \cap H$ of H. In turn, the positive systems define two orders $\leqslant$ and $\leqslant_H$ on X: set $\nu \leqslant \mu$ (resp., $\nu \leqslant_H \mu$) if $\mu - \nu \in \mathbb{Z}^+\Phi^+$ (resp., $\mu - \nu \in \mathbb{Z}^+\Phi^+_H$). We can also define the sets X^+ (resp., $X^{+(H)}$) of dominant (resp., H-dominant) weights as those elements $\lambda \in X$ which occur as the highest weights in an irreducible rational G-module $L(\lambda)$ (resp., irreducible rational H-module $L_H(\lambda)$). It will be useful to single out the dominant and H-dominant weights in any subset Λ of X. Thus, if $\Lambda \subseteq X$, we set $\Lambda^+ = \Lambda \cap X^+$ and $\Lambda^{+(H)} = \Lambda \cap X^{+(H)}$, respectively.

There is also a more "internal" description of the sets X^+ and $X^{+(H)}$: for a simple root α, let $\alpha^\vee : \mathbb{G}_m \to T$ be the corresponding coroot. Thus, $\alpha^\vee(t), t \in k^\times$, denotes the image of $\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right]$ under a map of the group $\mathrm{SL}_2(k)$ which sends the root group $\left[\begin{smallmatrix} 1 & * \\ 0 & 1\end{smallmatrix}\right]$ isomorphically to the root group U_α and sends the opposite root group of $\mathrm{SL}_2(k)$ to the opposite root group $U_{-\alpha}$. Then a weight $\nu \in X$ is dominant if and only if $\langle \nu, \alpha^\vee\rangle \geqslant 0$ for each $\alpha \in \Pi$. The weight ν belongs to $X^{+(H)}$ if $\langle \nu, \alpha^\vee\rangle \geqslant 0$ for all $\alpha \in \Pi_H = \Pi \cap \Phi^+_H$. The equivalence of this internal notion of dominance with the "external" one involving irreducible modules is well known for semisimple groups, and extends without difficulty to reductive groups. For more discussion, see [Jan3, II.2.6].

The next result follows easily from the fact that the Cartan matrix entry associated to distinct simple roots is always $\leqslant 0$.

Lemma 6. *Suppose $\mu \leqslant_H \lambda$ are weights in X, with $\mu \in X^{+(H)}$ and $\lambda \in X^+$. Then $\mu \in X^+$. In other words, X^+ is an ideal in the poset $(X^{+(H)}, \leqslant_H)$.*

Continuing with our notation, the categories of finite-dimensional rational G- and H-modules will be denoted $\mathcal{C}$ and $\mathcal{C}_H$, respectively. (Recall the convention at the start of Section 2.) Both $\mathcal{C}$ and $\mathcal{C}_H$ are highest weight categories with weight posets $(X^+, \leqslant)$ and $(X^{+(H)}, \leqslant_H)$, respectively. In the case of $\mathcal{C}$, the standard module $\Delta(\lambda)$, $\lambda \in X^+$, is the Weyl module with highest weight λ, while the costandard module $\nabla(\lambda)$ is the induced module $\mathrm{ind}_{B^-}^G \lambda$, where B^- is the Borel subgroup containing T and opposite to B and λ denotes the one-dimensional B^--module defined by inflating the character λ from T to B^-. Similar definitions apply to $\mathcal{C}_H$.

If $\Lambda \subseteq X$, it is useful (as suggested by previous notation) to write $\mathcal{C}[\Lambda]$ for the category of all modules $M \in \mathrm{Ob}(\mathcal{C})$ such that every composition factor $L(\nu)$ of M satisfies $\nu \in \Lambda$. Obviously, $\nu \in \Lambda^+$, so that $\mathcal{C}[\Lambda] = \mathcal{C}[\Lambda^+]$. Similarly, $\mathcal{C}_H[\Lambda]$ is defined and $\mathcal{C}_H[\Lambda] = \mathcal{C}_H[\Lambda^{+(H)}]$.

If $F \subseteq X$, we denote by X_F the ideal in the poset $(X, \leqslant)$ generated by F, i.e., $X_F = \{\, \gamma \in X \mid \gamma \leqslant \phi \text{ for some } \phi \in F \,\}$. If we regard a subset $Y \subseteq X$ as a subposet of X, then $Y_F := X_F \cap Y$ is an ideal in Y.

Now let $\Omega = \omega + \mathbb{Z}\Phi_H$ be a coset of $\mathbb{Z}\Phi_H$ in X. If $F \subseteq \Omega$, observe that $\Omega_F = \{\, \gamma \in \Omega \mid \gamma \leqslant_H \phi \text{ for some } \phi \in F \,\}$. Set $\Gamma = X_\Omega$, and for $F \subseteq \Omega$, observe that $\Gamma_F = \Gamma_{\Omega_F}$.

Let $F \subseteq \Omega$. At least when Γ_F is a finitely generated ideal in X (e.g., when F is finite), the category $\mathcal{C}[\Gamma_F] = \mathcal{C}[\Gamma_{\Omega_F}]$ is fully embedded in the full category of rational G-modules at the derived category level, cf. [CPS1, Theorem 3.9]. In particular, if M, N belong to $\mathcal{C}[\Gamma_F^+]$, then[2]

$$\mathrm{Ext}^\bullet_{\mathcal{C}[\Gamma_F^+]}(M, N) \cong \mathrm{Ext}^\bullet_G(M, N).$$

(See also [D2].) Obviously, if F is finite, then the set Γ_F^+ of dominant weights in Γ_F is also finite. In this way, the homological algebra of the category $\mathcal{C}[\Gamma] = \mathcal{C}[\Gamma^+]$ is completely determined by the various categories $\mathcal{C}[\Gamma_F] = \mathcal{C}[\Gamma_F^+]$, as F ranges over the finite subsets of Γ^+. Each $\mathcal{C}[\Gamma_F^+]$ is equivalent to the finite-dimensional module category of a quasi-hereditary algebra A_F [CPS1, Theorem 3.6]. Note that Ω_F^+ is a coideal of weights in Γ_F^+, yielding a recollement diagram

$$D^b(\mathcal{C}[\Gamma_F^+ \backslash \Omega_F^+]) \underset{\mathbf{i}^!}{\overset{\mathbf{i}^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathbf{i}_*}}}}} D^b(\mathcal{C}[\Gamma_F^+]) \underset{\mathbf{j}_*}{\overset{\mathbf{j}_!}{\underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathbf{j}^*}}}}} D^b(\mathcal{C}(\Omega_F^+)) \tag{7}$$

as in (1). Here $\mathcal{C}(\Omega_F^+)$ denotes the quotient category $\mathcal{C}[\Gamma_F^+]/\mathcal{C}[\Gamma_F^+ \backslash \Omega_F^+]$. As with (1), the above derived category functors are all induced from corresponding functors at the module category level, e.g., the quotient functor $\mathbf{j}^*$ is induced by a quotient functor $j^* : \mathcal{C}[\Gamma_F^+] \to \mathcal{C}(\Omega_F^+)$.

Next, we consider the truncation functor

$$\pi_\Omega : \mathcal{C}[\Gamma^+] \to \mathcal{C}_H[\Omega] = \mathcal{C}_H[\Omega^{+(H)}], \quad M \mapsto \pi_\Omega M = \textstyle\bigoplus_{\tau \in \Omega} M_\tau. \tag{8}$$

Here M_τ, $\tau \in \Omega$, denotes the τ-weight space of the rational G-module M.

The following property was observed in [D1] at the character-theoretic level, and at the module level in the irreducible case. Compare also [Sm]. Multiplicity results like Proposition 3.2(c) below seem to appear first in [Jan2] in a semisimple Lie algebra context. See especially Theorem 1.18 of that reference, attributed to Jantzen and Deodhar independently. The treatment in [Jan2] formalizes induction, but does not formalize truncation, as in [D1], though parts of the proof may be viewed as implicitly involving truncation.

[2]This isomorphism easily implies that, given finite-dimensional G-modules M, N, we have

$$\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N) \cong \mathrm{Ext}^\bullet_G(M, N),$$

where the former group is formed using n-fold Yoneda extensions.

Proposition 7. *Fix* $\lambda, \nu, \mu \in \Gamma^+$.

(a) *If* $\lambda \in \Omega^+$, *then*

$$\pi_\Omega \Delta(\lambda) \cong \Delta_H(\lambda), \pi_\Omega \nabla(\lambda) \cong \nabla_H(\lambda),\ \pi_\Omega L(\lambda) \cong L_H(\lambda).$$

(b) *If* $\nu \notin \Omega^+$, *then*

$$\pi_\Omega \Delta(\nu) \cong \pi_\Omega \nabla(\nu) \cong \pi_\Omega L(\nu) = 0.$$

(c) *If* $\mu \in \Omega^+$, *then the multiplicity of* $L(\mu)$ *as a composition factor of* $\Delta(\lambda)$ (*resp.,* $\nabla(\lambda)$) *equals the multiplicity of* $L_H(\mu)$ *as a composition factor of* $\Delta_H(\lambda)$ (*resp.,* $\nabla_H(\lambda)$).

Proof. To establish (a), we first observe that for $L(\lambda)$ and $\nabla(\lambda)$, $\lambda \in \Omega^+$, it is already proved in [Jan3, II.2.11] that $\pi_\Omega L(\lambda) \cong L_H(\lambda)$ and $\pi_\Omega \nabla(\lambda) \cong \nabla_H(\lambda)$. This easily implies the required assertion for $\Delta(\lambda)$, $\lambda \in \Omega$. In fact, since $\Delta(\lambda)$ (resp., $\Delta_H(\lambda)$) and $\nabla(\lambda)$ (resp., $\nabla_H(\lambda)$) have the same characters, $\pi_\Omega \Delta(\lambda)$ is a rational H-module with the same character and high weight as $\Delta_H(\lambda)$.[3] Clearly, $\pi_\Omega \Delta(\lambda)$ is generated as an H-module by a vector v of weight λ. (It is even generated by such a vector over the hyperalgebra of the negative Borel subgroup of H.) Thus, the map $\Delta_H(\lambda) \to \pi_\Omega \Delta(\lambda)$ obtained from the well known universal property of $\Delta_H(\lambda)$, sending a high weight vector of $\Delta_H(\lambda)$ to v, is an isomorphism. This proves (a).

If $\lambda, \nu \in \Gamma^+ \backslash \Omega^+$, then $\nu < \tau$ for some $\tau \in \Omega$. So $\tau - \nu$ must be a sum of positive roots, one of which is a simple root $\beta \notin \Phi_H$. Consequently, if $\theta < \nu$, then $\theta \notin \Omega$, because the difference $\tau - \theta$ must still involve a positive multiple of β. So $\pi_\Omega \Delta(\nu), \pi_\Omega \nabla(\nu), \pi_\Omega L(\nu)$ must all be zero, and (b) holds.

Finally, observe that if $L(\mu)$ is a composition factor of either $\Delta(\lambda)$ or $\nabla(\lambda)$, then $\mu \in \Gamma^+$. Thus (c) follows from exactness of π_Ω, together with (a) and (b). □

Before stating the next result, we draw several conclusions about the various posets involved. First, for any subset $F \subseteq \Omega$, Ω_F^+ is a coideal in the poset $(\Gamma_F^+, \leqslant)$. In turn, Γ_F^+ is an ideal in $(X^+, \leqslant)$. Suppose now that F is finite. Thus, the highest weight category $\mathcal{C}(\Omega_F) := \mathcal{C}[\Gamma_F^+]/\mathcal{C}[\Gamma_F^+ \backslash \Omega_F^+]$ with weight poset $(\Omega_F^+, \leqslant) = (\Omega_F^+, \leqslant_H)$ is defined [CPS1, Theorem 3.5]. On the other hand, Lemma 6 implies that Ω_F^+ is an *ideal* in the poset $(X^{+(H)}, \leqslant_H)$. Thus, the highest weight category $\mathcal{C}_H[\Omega_F^+]$ with weight poset $(\Omega_F^+, \leqslant_H) = (\Omega_F^+, \leqslant)$ is defined. Similarly, Ω^+ is a coideal in $(\Gamma^+, \leqslant)$, but an ideal in $(X^{+(H)}, \leqslant_H)$. We form the quotient category $\mathcal{C}(\Omega^+) := \mathcal{C}[\Gamma^+]/\mathcal{C}[\Gamma^+ \backslash \Omega^+]$; see the discussion on quotient categories given in [F, Ch. 15]. Then the irreducible objects in the categories $\mathcal{C}(\Omega^+)$ and $\mathcal{C}_H[\Omega^+]$ are both indexed by the same poset $(\Omega^+, \leqslant) = (\Omega^+, \leqslant_H)$. The category $\mathcal{C}_H[\Omega^+]$ is a highest weight category, and we will prove that $\mathcal{C}(\Omega^+)$ is equivalent to it.[4]

[3]Alternatively, without appealing to [Jan3], this fact follows formally from the observation that the character of $\Delta(\lambda)$ is the alternating sum over $w \in W$ (the Weyl group) of the characters of the Verma modules (for the complex Lie algebra having the same root type as G) of high weight $w \cdot 0$. Similar remarks apply to $\Delta_H(\lambda)$. Taken with the remainder of this paragraph (and the dual argument), the results cited in [Jan3] can be reproved.

[4]In case Ω^+ and Γ^+ are finite, the discussion of [CPS1] implies that $\mathcal{C}(\Omega^+)$ is a highest weight category. This case is close to the situation discussed in the first paragraph of §2.2 for a highest weight category A-mod. In general, the finite generation hypotheses of [CPS1, Theorem 3.9] need not hold, so it is *a priori* unclear if the quotient category $\mathcal{C}(\Omega^+)$ is a highest weight category.

Theorem 8. *If $F \subseteq \Omega^+$ is finite, the functor π_Ω, restricted to the category $\mathcal{C}[\Gamma^+_F]$, factors through the quotient morphism $j^* : \mathcal{C}[\Gamma^+_F] \to \mathcal{C}(\Omega^+_F)$ to produce an equivalence*

$$\mathcal{C}(\Omega^+_F) \xrightarrow[\sim]{\overline{\pi_\Omega}} \mathcal{C}_H[\Omega^+_F] \tag{9}$$

of categories. Similarly, the functor π_Ω, restricted to the subcategory $\mathcal{C}[\Gamma^+]$, factors through the quotient morphism $j^* : \mathcal{C}[\Gamma^+] \to \mathcal{C}(\Omega^+)$ to produce an equivalence

$$\mathcal{C}(\Omega^+) \xrightarrow[\sim]{\overline{\pi_\Omega}} \mathcal{C}_H[\Omega^+] \tag{10}$$

of categories. In particular, $\mathcal{C}_H(\Omega^+)$ is a highest weight category.

Proof. First, assume that (9) has been proved. Taking directed unions, as F ranges through all finite subsets of Ω^+, we also obtain an equivalence

$$\overline{\pi_\Omega} : \mathcal{C}(\Omega^+) \xrightarrow{\sim} \mathcal{C}_H[\Omega^+],$$

so that (10) holds.

Thus, it suffices to prove (9). Since F is finite, $\mathcal{C}_H[\Omega^+_F]$ is a highest weight category with finitely many simple objects, the irreducible modules $L_H(\nu)$ with $\nu \in \Omega^+_F$. Of course, $\mathcal{C}(\Omega^+_F)$ is a highest weight category with the same finite weight poset, and we also note that its standard and costandard objects are inherited from $\mathcal{C}[\Gamma^+_F]$. Clearly, the functor π_Ω, applied to the category $\mathcal{C}[\Gamma^+_F]$, factors through the quotient category $\mathcal{C}(\Omega^+_F)$, as may be observed by its behavior on irreducible objects. Thus, we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{C}[\Gamma^+_F] & \xrightarrow{\quad j^* \quad} & \mathcal{C}(\Omega^+_F) \\ & {\scriptstyle \pi_\Omega} \searrow \quad \swarrow {\scriptstyle \overline{\pi_\Omega}} & \\ & \mathcal{C}_H[\Omega^+_F] & \end{array} .$$

The quotient morphism j^* admits a left adjoint $j_!$ and a right adjoint j_*, as discussed in §2.2. Both $j_!$ and j_* are sections for j^*, while $j_!$ is right exact and j_* is left exact. Since

$$\overline{\pi_\Omega} = \overline{\pi_\Omega}\mathrm{id}_{\mathcal{C}(\Omega^+_F)} = \overline{\pi_\Omega}\mathrm{j}^*\mathrm{j}_! = \pi_\Omega \mathrm{j}_!,$$

the exactness of π_Ω and the right exactness of $j_!$ imply that $\overline{\pi_\Omega}$ is right exact. A similar calculation shows that $\overline{\pi_\Omega}$ is left exact. Therefore, $\overline{\pi_\Omega}$ is an exact functor. But now, using Proposition 7, the Comparison Theorem 2 applies, giving an equivalence

$$\overline{\pi_\Omega} : \mathcal{C}(\Omega^+_F) \xrightarrow{\sim} \mathcal{C}_H[\Omega^+_F].$$

This completes the proof of the theorem. □

It often turns out that Ω^+ is itself finite. The proof of the proposition below relies on the fact that each entry in the inverse of the Cartan matrix of a simple algebraic group is a positive rational number. For a nice exposition of this fact, see [LT].

Proposition 9. *Assume that H is a* Levi *subgroup of a proper parabolic subgroup P of the reductive group G. Assume that no simple component of Φ_H is a component of Φ. (For example, this last condition holds if G, or its commutator subgroup G', is simple.) If $\Omega = \omega + \mathbb{Z}\Phi_H$ is a coset of $\mathbb{Z}\Phi_H$ in X, then the set Ω^+ is finite. In this case, $\Gamma^+ = X^+_\Omega$ is also finite.*

Proof. The proof easily reduces to the case when Φ is connected and $G = G'$. We give the argument in this case. We may assume Ω^+ is non-empty and choose the coset representative ω to be a dominant weight. Since H is proper, there is a simple root α in $\Pi \setminus \Pi_H$. Write ω as a rational linear combination of simple roots, and let a denote the coefficient of α in this expression. Since any other element μ of Ω differs from ω by an integral linear combination of the roots in Π_H, the coefficient of α in the analogous expression for μ is also equal to a.

Let $\{\varpi_\alpha\}_{\alpha\in\Pi} \subset X$ be the fundamental dominant weights corresponding to Φ, i.e., $\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$, $\alpha, \beta \in \Pi$. If we assume $\mu = \sum_{\beta\in\Pi} m_\beta \beta$ is dominant, then we have

$$a = \sum_{\beta\in\Pi} m_\beta r_{\beta,\alpha},$$

where $m_\beta \geqslant 0$ is an integer and the coefficients $r_{\beta,\alpha}$ are positive rational numbers. This equation clearly has only finitely many solutions. Thus, Ω^+ is finite.

Given $\gamma \in \Gamma^+$, $\gamma \leqslant \omega$ for some $\omega \in \Omega$. Since Ω is W_H-stable, $\omega \leqslant \omega' \in \Omega^+$. Therefore, $\gamma \leqslant \omega' \in \Omega^+$. The finiteness of Ω^+ implies that Γ^+ is finite. □

The category equivalences of the theorem, when coupled with the derived category recollement diagrams for the various $\mathcal{C}(\Omega^+_F)$, are quite powerful. We have the following Ext-transfer result, using Lemma 3:

Corollary 10. *If λ, μ are dominant weights in Ω, then*

$$\mathrm{Ext}^\bullet_G(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^\bullet_H(\Delta_H(\lambda), L_H(\mu))$$

and

$$\mathrm{Ext}^\bullet_G(L(\mu), \nabla(\lambda)) \cong \mathrm{Ext}^\bullet_H(L_H(\mu), \nabla_H(\lambda)).$$

Proof. We just prove the first assertion; the second follows similarly. Let $\lambda, \mu \in \Omega^+$. Choose some finite subset $F \subset \Omega$ such that $\lambda, \mu \in \Omega^+_F$. (It is not difficult to see that one could work with $F = \Omega$.) By Lemma 3,

$$\Delta(\lambda) \cong \mathbf{j}_! \Delta_{\Omega^+_F}(\lambda) = j_! \Delta_{\Omega^+_F}(\lambda),$$

using the recollement diagram (7). Thus, by Theorem 8,

$$\begin{aligned}
\mathrm{Ext}^\bullet_G(\Delta(\lambda), L(\lambda)) &\cong \mathrm{Ext}^\bullet_{\mathcal{C}[\Gamma^+_F]}(\Delta(\lambda), L(\mu)) \\
&\cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C}[\Gamma^+_F])}(\Delta(\lambda), L(\mu)) \\
&\cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C}[\Gamma^+_F])}(\mathbf{j}_! \Delta_{\Omega^+_F}(\lambda), L(\mu)) \\
&\cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C}(\Omega^+_F))}(\Delta_{\Omega^+_F}(\lambda), \mathbf{j}^* L(\mu)) \\
&\cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C}(\Omega^+_F))}(\Delta_{\Omega^+_F}(\lambda), L_{\Omega^+_F}(\mu)) \\
&\cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C}[\Omega^+_F])}(\overline{\pi_\Omega} \Delta_{\Omega^+_F}(\lambda), \overline{\pi_\Omega} L_{\Omega^+_F}(\mu)) \\
&\cong \mathrm{Ext}^\bullet_H(\Delta_H(\lambda), L_H(\mu)),
\end{aligned}$$

as required. □

Remarks 11. (a) If $p \geqslant h$ (the Coxeter number of G) and the modular Lusztig conjecture holds for G, then the dimensions of $\mathrm{Ext}^n_G(\Delta(\lambda), L(\mu))$ and $\mathrm{Ext}^n_G(L(\mu), \nabla(\lambda))$ can be determined, at least when λ, μ are p-regular, in terms of the coefficients of Kazhdan–Lusztig polynomials for the affine Weyl group W_p associated to G [A1, (2.1.2)]. From these dimensions, the dimensions of the groups $\mathrm{Ext}^n_G(L(\lambda), L(\mu))$ can also be determined, using [CPS3, (3.9.1)]. In some cases, when p is too small or λ, μ are not p-regular, Ext-calculations can be reduced, using the above Ext-transfer, to similar calculations for Levi subgroups H where the above hypotheses are satisfied. An illustration of this kind of procedure, in type $\mathtt{A}$, is given in [PS2, §6].

(b) As pointed out to us by one referee, one can also establish Corollary 10 directly. We sketch the details. Let $P^- = H \ltimes U_{P^-}$ be the parabolic subgroup with Levi factor H which contains the negative Borel subgroup B^-. Then $\mathrm{Ext}^\bullet_G(L(\mu), \nabla(\lambda)) \cong \mathrm{Ext}^\bullet_G(L(\mu), \mathrm{Ind}^G_{P^-}\nabla_H(\lambda)) \cong \mathrm{Ext}^\bullet_{P^-}(L(\mu), \nabla_H(\lambda))$, where $\nabla_H(\lambda)$ is regarded as a rational P^--module through inflation. In this expression, the second isomorphism follows easily from [CPSK, Theorem 3.1], applied to the module denoted there by $M = \nabla_H(\lambda)$. (One needs that $R^n\mathrm{Ind}^G_{P^-}\nabla_H(\lambda) = 0$ for $n > 0$, but this follows immediately from the Grothendieck spectral sequence corresponding to the functor factorization $\mathrm{Ind}^G_{B^-} = \mathrm{Ind}^G_{P^-} \circ \mathrm{Ind}^{P^-}_{B^-}$ together with Kempf's theorem [Jan3, II.4.6] for G and for P^-.) Consider the Hochschild–Serre spectral sequence

$$E_2^{p,q} = H^p(H, H^q(U, L(\mu)^* \otimes \nabla_H(\lambda))) \Rightarrow \mathrm{Ext}^{p+q}_{P^-}(L(\mu), \nabla_H(\lambda)).$$

As a rational H-module, $L(\mu) \cong L_H(\mu) \oplus D$, where the composition factors $L_H(\xi)$ of D satisfy $\xi \notin \Omega$ as well as $\xi < \mu$. If $q > 0$, the weights of T in $H^q(U, L_H(\mu)^* \otimes \nabla_H(\lambda)))$ have the form $\xi + \theta$, where ξ is a nontrivial sum of roots in $\Phi^+\backslash\Phi^+_H$ and θ lies in $\mathbb{Z}\Phi_H$. Hence, $H^p(H, H^q(U, L_H(\mu)^* \otimes \nabla_H(\lambda)) = 0$ if $q > 0$. (Here we use the fact that the weights in $H^q(U, k)$, $q > 0$, are all nontrivial sums of roots in $\Phi^+\backslash\Phi^+_H$; see [CPSK, §2].) Next, observe that for all $q \geqslant 0$, $H^p(H, H^q(U, D \otimes \nabla_H(\lambda))) = 0$. Otherwise, there exists weights ζ, ξ, and σ with the following properties: (1) ζ is a nontrivial sum of positive roots, not all in Φ^+_H; (2) ξ is a sum, possibly empty, of roots in $\Phi^+\backslash\Phi^+_H$; (3) $\sigma \in \mathbb{Z}\Phi_H$; and (4) $\xi + (-\mu + \zeta) + (\lambda - \sigma) \in \mathbb{Z}\Phi_H$. (Here $-\mu + \zeta$ corresponds to a weight in D and $\lambda - \sigma$ to a weight in $\nabla_H(\lambda)$.) We use the fact that, given a rational H-module M, if $H^q(H, M) \neq 0$, then some weight of M must lie in $\mathbb{Z}\Phi_H$ [CPSK, Theorem 2.4]. Equation (4) is clearly impossible since $\lambda - \mu \in \mathbb{Z}\Phi_H$. Finally, these facts together imply that the spectral sequence collapses to the required $\mathrm{Ext}^\bullet_H(L_H(\mu), \nabla_H(\lambda))$.

Although the above argument is more direct, the original proof of Corollary 10, based on the categorical equivalence of Theorem 8, remains more conceptual. In addition, it should be mentioned that the existence of Theorem 8 led to the discovery of Corollary 10.

(c) Anton Cox has pointed out to us a result of Donkin and Erdmann, given in [E1, (4.3)], which is similar to the first part of Corollary 10, but with $L(\mu)$ replaced with $\Delta(\mu)$. Such a result is in the spirit of the original results in [C], especially [C, Corollary 6.7]. Specifically, the result states, under the same hypotheses as in Corollary 10, that

$$\mathrm{Ext}^\bullet_G(\Delta(\lambda), \Delta(\mu)) \cong \mathrm{Ext}^\bullet_H(\Delta_H(\lambda), \Delta_H(\mu)).$$

Using the methods of this paper, a proof of this result is obtained from that for Corollary 10 by replacing $L(\mu)$ by $\Delta(\mu)$ and using the fact, mentioned in (2), that shows

$j^*\Delta(\mu) \cong \Delta_H(\mu)$. Dually, one has

$$\mathrm{Ext}^{\bullet}_G(\nabla(\lambda), \nabla(\mu)) \cong \mathrm{Ext}^{\bullet}_H(\nabla_H(\lambda), \nabla_H(\mu)).$$

As in the case of Corollary 10 itself, a direct proof of this isomorphism can be based on the argument given in (b); this is, in fact, similar to the argument in [E1].

(d) We finally observe that Theorem 8 trivially holds when k is a field of characteristic 0. In this case, all the highest weight categories involved are semisimple. One referee has raised the question of the validity of the theorem for reductive group schemes over $\mathbb{Z}$ or over the localization $\mathbb{Z}_{(p)}$. This seems likely to us, but we have not pursued it.

Using an Euler characteristic argument, Corollary 10 gives an interesting interpretation of the multiplicity identification [D1]. See also [Jan3, II.5.21].

Corollary 12. *Let λ, μ be dominant weights in Ω. Then*

$$[\Delta(\lambda) : L(\mu)] = [\Delta_H(\lambda) : L_H(\mu)].$$

Proof. If $M \in \mathcal{C}$, let $[M]$ denote its image in the Grothendieck group of $\mathcal{C}$. Use the same notation to denote objects in the Grothendieck group of $\mathcal{C}_H$.

For $\nu, \tau \in X^+$, write

$$[L(\nu)] = \textstyle\sum_\sigma p_{\sigma,\nu}[\Delta(\sigma)]. \tag{11}$$

This is certainly possible, using only the finitely many $\sigma \leqslant \nu$, by a well known argument with unitriangular matrices. Also, $p_{\nu,\nu} = 1$. By a formal result ("Delorme's theorem"—see [CPS3, (3.2)]),

$$p_{\sigma,\nu} = \textstyle\sum_i (-1)^i \dim \mathrm{Ext}^i_G(L(\nu), \nabla(\sigma)). \tag{12}$$

Similarly, we have

$$[L_H(\nu)] = \textstyle\sum_\sigma p^H_{\sigma,\nu}[\Delta_H(\sigma)], \tag{13}$$

where

$$p^H_{\sigma,\nu} = \textstyle\sum_i (-1)^i \dim \mathrm{Ext}^i_H(L_H(\nu), \nabla_H(\sigma)).$$

The previous Corollary 10 shows that if σ, ν both belong to Ω, then

$$p_{\sigma,\nu} = p^H_{\sigma,\nu}. \tag{14}$$

Write

$$[\Delta(\lambda)] = \textstyle\sum_\tau q_{\tau,\lambda}[L(\tau)]; \quad [\Delta_H(\lambda)] = \sum_\xi q^H_{\xi,\lambda}[L_H(\xi)]. \tag{15}$$

We must show that $q_{\mu,\lambda} = q^H_{\mu,\lambda}$. But if we substitute (15) into (11), we find that the q's are, as expected, "inverse" to the p's:

$$\textstyle\sum_\nu p_{\tau,\nu} q_{\nu,\lambda} = \delta_{\tau,\lambda}, \quad \sum_\nu p^H_{\tau,\nu} q^H_{\nu,\lambda} = \delta_{\tau,\lambda}.$$

Now (14) easily yields, recursively, the desired equalities. □

4. Schur algebras, I: first Ext-transfer

In this section, we situate the theory of Schur algebras in the setting of the previous section.

We first recall some notation, some of it already used. For a positive integer r, let $\Lambda^+(r)$ be the set of all partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0)$ of r. It will be convenient to regard λ as an infinite non-increasing sequence of non-negative integers, eventually equal to 0. Given $\lambda \in \Lambda^+(r)$, the dual partition λ' is defined by setting $\lambda'_i = \#\{\lambda_j \mid \lambda_j \geqslant i\}$. For $\lambda, \mu \in \Lambda^+(r)$, put $\lambda \trianglelefteq \mu$ provided that $\lambda_1 + \cdots + \lambda_i \leqslant \mu_1 + \cdots + \mu_i$ for all i. In this way, $\Lambda^+(r)$ is a poset. For a positive integer n, the subset $\Lambda^+(n,r)$ of $\Lambda^+(r)$ consisting of all partitions λ with at most n nonzero parts is a coideal in $\Lambda^+(r)$. Clearly, $\lambda \trianglelefteq \mu \iff \mu' \trianglelefteq \lambda'$.

Given $\lambda \in \Lambda^+(r_1)$ and $\mu \in \Lambda^+(r_2)$, define $\lambda + \mu \in \Lambda^+(r_1 + r_2)$ by setting $(\lambda+\mu)_i = \lambda_i + \mu_i$ for all i.

Let k be an algebraically closed field of characteristic $p \geqslant 0$. Put $V = k^n$, the space of $n \times 1$ column vectors, and, given a positive integer r, let $V^{\otimes r}$ be the space of r-tensors. The symmetric group $\mathfrak{S}_r$ of degree r acts on $V^{\otimes r}$ on the right by place displacement: $(v_1 \otimes \cdots \otimes v_r)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(1)}$, for $v_1, \cdots, v_n \in V$ and $\sigma \in \mathfrak{S}_r$. The Schur algebra $S(n,r)$ is defined to be the endomorphism algebra $\mathrm{End}_{k\mathfrak{S}_r}(V^{\otimes r})$ for this action of $\mathfrak{S}_r$ on $V^{\otimes r}$. Because the action of $\mathrm{GL}_n(k)$ on V extends to an action of $\mathrm{GL}_n(k)$ on $V^{\otimes r}$, there is a natural homomorphism $\xi : \mathrm{GL}_n(k) \to \mathrm{GL}(V^{\otimes r})$. This (left) action of $\mathrm{GL}_n(k)$ commutes with the action of $\mathfrak{S}_r$, and $S(n,r)$ identifies with the subalgebra of $\mathrm{End}(V^{\otimes r})$ generated by the image of ξ. For further details on the general theory of Schur algebras, see [Gr, Ch. 2]. Other references on Schur algebras which are relevant to this paper are [D4] and [P].

It is well known that the module category $S(n,r)$-mod is a highest weight category with respect to the poset $\Lambda^+(n,r)$ described above. We can also identify $S(n,r)$-mod as the category of polynomial representations of $\mathrm{GL}_n(k)$ which are homogeneous of degree r. In addition, for $M, N \in S(n,r)$-mod,

$$\mathrm{Ext}^\bullet_{S(n,r)}(M,N) \cong \mathrm{Ext}^\bullet_{\mathrm{GL}_n(k)}(M,N). \tag{16}$$

(See [D2] or the discussion in §3.)

Consider the reductive group $\mathrm{GL}_r(k)$. For $0 < n < r$, form the Levi subgroup $H = \mathrm{GL}_n(k) \times \mathbb{G}_m^{r-n}$ of $\mathrm{GL}_r(k)$. Explicitly, H consists of matrices $\left[\begin{smallmatrix} A & 0 \\ 0 & t \end{smallmatrix}\right]$ with $A \in \mathrm{GL}_n(k), t \in \mathbb{G}_m^{r-n}$.[5] The character group X of the maximal torus T of diagonal matrices identifies naturally with sequences $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers; we call $|\lambda| := \lambda_1 + \cdots + \lambda_r$ the *degree* of λ. The root system Φ of T on the Lie algebra $\mathrm{Lie}(\mathrm{GL}_r(k))$ consists of those sequences λ of weight $|\lambda| = 0$ which have exactly two nonzero entries, one of them equal to $+1$. Then $\alpha = (a_1, \ldots, a_r) \in \Phi$ is positive if the first nonzero a_i equals $+1$. For $\lambda, \mu \in X$, put $\lambda \geqslant \mu$ provided $\lambda - \mu$ is a sum of positive roots. If $\lambda, \mu \in \Lambda^+(r)$, then $\lambda \leqslant \mu \iff \lambda \trianglelefteq \mu$. Also, $\alpha \in \Phi$ belongs to the root system Φ_H of T in $\mathrm{Lie}(H)$ if and only if $a_i = 0$ for $i > n$.

[5] In the proof of Theorem 13, we will also use the Levi subgroup $\mathbb{G}_m^{r-n} \times \mathrm{GL}_n(k)$ consisting of matrices $\left[\begin{smallmatrix} t & 0 \\ 0 & A \end{smallmatrix}\right]$, $A \in \mathrm{GL}_n(k), t \in \mathbb{G}_m^{r-n}$. The reader can immediately modify the discussion below to include this case.

The irreducible $\mathrm{GL}_r(k)$-modules are indexed by the subset X^+ of X consisting of non-increasing sequences $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. In turn, the irreducible H-modules are indexed by the set $X^{+(H)} \subset X$ of sequences $(\mu_1, \ldots, \mu_n, \ldots, \mu_r)$ of integers in which $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$.

Let $\omega = (r, 0, \ldots, 0) \in X^{+(H)} \cap X^+$ and form the coset $\Omega = \omega + \mathbb{Z}\Phi_H$. Let $F = \{\omega\}$. Then $\Omega_F^+ = \Lambda^+(n,r)$ is an ideal in $X^{+(H)}$ and a coideal in $\Gamma_F^+ = \Lambda^+(r)$. The category $\mathcal{C}[\Gamma_F^+]$ is equivalent to the category $S(r,r)$-mod. It is well known that $S(n,r)$-mod is the quotient category of $S(r,r)$ by the Serre subcategory $S(r,r)\text{-mod}[\Lambda^+(r)\backslash\Lambda^+(n,r)]$. Thus, Corollary 10 identifies $S(n,r)$-mod with the highest weight category $\mathcal{C}_H[\Omega_F^+]$ associated to the Levi subgroup H.

In this case, the Ext-identifications in Corollary 10 coincide with those obtainable from the more direct quotient category viewpoint for $S(n,r)$; see [PS2, Theorem 6.1], which in turn was based on Lemma 3. However, we can obtain more information, as we now describe.

Theorem 13. *Let $\lambda, \mu \in \Lambda^+(r)$. Suppose that $\widetilde{\lambda}$, $\widetilde{\mu}$ are obtained from λ, μ by one of the following processes* (*or any iteration of them*):

(i) *removal or addition of equal first rows;*
(ii) *removal or addition of equal last rows, when the total number of nonzero rows is the same;*
(iii) *removal or addition of equal first columns;*
(iv) *removal or addition of equal last columns, when the total number of nonzero columns is the same.*

Here we have identified the partitions with their Young diagrams in the usual way.

Let $\widetilde{r}$ denote the common degree $|\widetilde{\lambda}| = |\widetilde{\mu}|$. Then

$$\begin{aligned}\mathrm{Ext}^\bullet_{S(r,r)}(\Delta(\lambda), L(\mu)) &\cong \mathrm{Ext}^\bullet_{S(\widetilde{r},\widetilde{r})}(\Delta(\widetilde{\lambda}), L(\widetilde{\mu})),\\ \mathrm{Ext}^\bullet_{S(r,r)}(L(\lambda), \nabla(\mu)) &\cong \mathrm{Ext}^\bullet_{S(\widetilde{r},\widetilde{r})}(L(\widetilde{\lambda}), \nabla(\widetilde{\mu})).\end{aligned}$$

Proof. Notice that part (iv) follows from part (i), and part (ii) follows from part (iii). Also, it is enough in each case to treat only the "removal" alternative. For part (iii), let $\widetilde{\lambda}, \widetilde{\mu}$ be obtained by removal of an equal first column from λ, μ. Then $\Delta(\widetilde{\lambda})$, $L(\widetilde{\mu})$ have the same restrictions to $\mathrm{SL}_n(k)$ as do $\Delta(\lambda)$, $L(\lambda)$, respectively, where n is the length of the common first column of λ, μ. However, if M, N are homogeneous modules of $\mathrm{GL}_n(k)$ with the same degree of homogeneity, then $\mathrm{Ext}^n_{\mathrm{GL}_n(k)}(M,N) \cong \mathrm{Ext}^n_{\mathrm{SL}_n(k)}(M,N)$. Thus, using (16),

$$\mathrm{Ext}^\bullet_{S(n,r)}(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^\bullet_{\mathrm{SL}_n(k)}(\Delta(\lambda), L(\mu))$$

and a similar statement holds for $\widetilde{r}$ in place of r (even if $n > \widetilde{r}$). The desired result now follows from Corollary 10 or [PS2, Theorem 6.1].

For Part (i), first note that λ, μ each have at most $m = 1 + \widetilde{r}$ parts. Then we have, by using (16) again,

$$\mathrm{Ext}^\bullet_{S(r,r)}(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^\bullet_{S(m,r)}(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^\bullet_{\mathrm{GL}_m(k)}(\Delta(\lambda), L(\mu)).$$

This is a slight abuse of notation, since the meaning of $\Delta(\lambda), L(\mu)$ remains the same, although the objects themselves and categories they are in varies. Let $G = \mathrm{GL}_m(k)$, and let H in Corollary 10 be the group $\mathbb{G}_m \times \mathrm{GL}_{\widetilde{r}}(k)$, and set $\widetilde{H} \cong \mathrm{GL}_{\widetilde{r}}(k)$ equal to the subgroup of all invertible matrices $\left[\begin{smallmatrix}1 & 0\\ 0 & A\end{smallmatrix}\right]$ where A is an $\widetilde{r} \times \widetilde{r}$ matrix. Clearly, $\lambda - \mu = \widetilde{\lambda} - \widetilde{\mu}$ as weights for the full diagonal subgroup of $\mathrm{GL}_m(k)$. We denote the common difference by η. If η is not in the root lattice for $\mathrm{GL}_n(k)$, then

$$\mathrm{Ext}^\bullet_G(\Delta(\eta), L(\mu)) \cong \mathrm{Ext}^\bullet_H(\Delta_H(\lambda), L_H(\mu)) = 0.$$

However, if η *is* in this root lattice, it is also in the root lattice for $\widetilde{H}$, and, thus, of H. Hence, Corollary 10 applies, and we have

$$\mathrm{Ext}^\bullet_G(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^\bullet_H(\Delta_H(\lambda), L_H(\mu)).$$

However, it is obvious that $\Delta_H(\lambda)$, $L_H(\mu)$ are obtained from their counterparts, $\Delta_{\widetilde{H}}(\widetilde{\lambda})$, $L_{\widetilde{H}}(\widetilde{\mu})$ on $\widetilde{H}$ by just tensoring (externally) with a 1-dimensional module in which $\mathbb{G}_m$ acts through the homomorphism $t \mapsto t^{r-\widetilde{r}}$. Part (iii) is now a consequence of the Künneth formula.

An alternative proof of part (ii) may be obtained by similarly replacing $H = \mathbb{G}_m \times \widetilde{H}$ with the Levi subgroup (in the notation above the statement of the theorem) $\widetilde{H} \times \mathbb{G}_m$, with $r - \widetilde{r}$ here equal to the common length of the bottom rows of λ, μ. (This is how part (ii), and, then, part (iv) were discovered.) □

Remarks 14. (a) Similar results hold for decomposition numbers, as observed in [D1], for cases (i) and (iii). (Case (ii) works as well for decomposition numbers.) That is, if λ, μ are related to $\widetilde{\lambda}, \widetilde{\mu}$ as hypothesized, then

$$[\Delta(\lambda) : L(\mu)] = [\Delta(\widetilde{\lambda}) : L(\widetilde{\mu})].$$

(b) In the spirit of Remark 11(b), we have, using the hypotheses and notation of Theorem 13,

$$\begin{aligned}\mathrm{Ext}^\bullet_{S(r,r)}(\Delta(\lambda), \Delta(\mu)) &\cong \mathrm{Ext}^\bullet_{S(\widetilde{r},\widetilde{r})}(\Delta(\widetilde{\lambda}), \Delta(\widetilde{\mu})),\\ \mathrm{Ext}^\bullet_{S(r,r)}(\nabla(\lambda), \nabla(\mu)) &\cong \mathrm{Ext}^\bullet_{S(\widetilde{r},\widetilde{r})}(\nabla(\widetilde{\lambda}), \nabla(\widetilde{\mu})).\end{aligned}$$

The proof is an evident modification of that of Theorem 13.

5. Schur algebras, II: Second Ext-transfer

We will be considering Schur algebras $S(n, r)$ for various choices of positive integers n, r. We assume that k has positive characteristic p. Let $F : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$, $g = [a_{i,j}] \mapsto g^F = [a^p_{i,j}]$, be the Frobenius morphism. Set $\mathrm{GL}_n(k)_1 = \mathrm{Ker}(F)$, the scheme-theoretic kernel of F. Given a rational $\mathrm{GL}_n(k)$-module V, $V^{(1)}$ denotes the rational $\mathrm{GL}_n(k)$-module obtained from V by making $\mathrm{GL}_n(k)$ act through F. Let $\rho_n = (n-1, n-2, \dots, 1, 0) \in \Lambda^+(n, n(n-1)/2)$ and let $\mathrm{St} = L((p-1)\rho_n) \in S(n, (p-1)n(n-1)/2)$-mod be the Steinberg module. As a module for the infinitesimal subgroup $\mathrm{GL}_n(k)_1$, St is projective and irreducible. For $\lambda \in \Lambda^+(n, r)$, let, as usual, $P(\lambda) \in S(n, r)$-mod be the indecomposable projective module with head $L(\lambda)$.

The following lemma is well known, and follows easily from the Steinberg tensor product theorem and the Hochschild-Serre spectral sequence for the pair $(\mathrm{GL}_r(k), \mathrm{GL}_r(k)_1)$. (For example, see [Jan3, Proposition II.3.19] for a proof of (a).) Recall from §3 that, given $\lambda \in \Lambda^+(r, r)$, $Y(\lambda)$ denotes the indecomposable tilting module for $S(r, r)$ of high weight λ.

Lemma 15. *Let* $\lambda \in \Lambda^+(r,r)$ *and set* $s = pr + (p-1)r(r-1)/2$. *Then:*

(a) $\Delta(\lambda)^{(1)} \otimes \mathrm{St} \cong \Delta(p\lambda + (p-1)\rho_r) \in S(r,s)$-mod;
(b) $P(\lambda)^{(1)} \otimes \mathrm{St} \cong P(p\lambda + (p-1)\rho_r) \in S(r,s)$-mod;
(c) $Y(\lambda)^{(1)} \otimes \mathrm{St} \cong Y(p\lambda + (p-1)\rho_r) \in S(r,s)$-mod.

Define a function

$$d : \Lambda^+(r) \to \Lambda^+(r,s), \ \lambda \mapsto (p\lambda' + (p-1)\rho_r)'. \tag{17}$$

We will consider the left exact, additive functor

$$\mathfrak{J} : S(r,r)\text{−mod} \to S(s,s)\text{-mod}[\Lambda^+(r,s)'] \tag{18}$$

defined as the composition

$$\mathfrak{J} = G_2 \circ T_2 \circ F_1 \circ G_1 \circ T_1$$

of five functors given as follows. First, $T_1 = T_{S(r,r)} : S(r,r)\text{-mod} \to E(r,r)\text{-mod}$ is the tilting functor (5), while $G_1 : E(r,r)\text{-mod} \to S(r,r)\text{-mod}$ is the category equivalence described in Theorem 5(a) when $q = 1$. Next, $F_1 : S(r,r)\text{-mod} \to S(r,s)\text{-mod}$ is the exact functor defined by $F_1(M) = M^{(1)} \otimes \mathrm{St}$ for all $M \in S(r,r)$-mod, while T_2 is the tilting functor $T_{S(r,s)}$, and G_2 is the category equivalence given in Theorem 5(b) for $E_q(r,s)$, when $q = 1$. For $\lambda \in \Lambda^+(r)$, we have

$$\mathfrak{J}\Delta(\lambda) \cong \Delta(d(\lambda)), \ \mathfrak{J}P(\lambda) \cong P(d(\lambda)), \tag{19}$$

where $\Delta(d(\lambda))$ and $P(d(\lambda))$ are standard and indecomposable projective modules in the highest weight category $S(s,s)\text{-mod}[\Lambda(r,s)']$. (Observe that $\Delta(d(\lambda))$ is a standard module in $S(s,s)$-mod, but $P(d(\lambda))$ is *not* necessarily a projective module in $S(s,s)$-mod.)

Suppose that Γ is an ideal in the poset $\Lambda^+(a,b)$. In the following proof, we will use the fact that

$$[\Delta(\lambda) : L(\mu)] = [P(\mu) : \Delta(\lambda)], \quad \forall \lambda, \mu \in \Gamma$$

in the category $S(a,b)\text{-mod}[\Gamma]$. This follows from Brauer–Humphreys reciprocity [CPS3, (3.11)], plus the fact that $S(a,b)\text{-mod}[\Gamma]$ admits a strong duality D.[6]

Lemma 16. *The image of* $d : \Lambda^+(r) \to \Lambda^+(s)$ *is a union of blocks for the highest weight category* $S(s,s)\text{-mod}[\Lambda^+(r,s)']$.

Proof. Suppose that $d(\lambda)$ and $\mu \in \Lambda^+(r,s)'$ belong to the same block for the category $S(s,s)\text{-mod}[\Lambda^+(r,s)']$. (That is, $L(d(\lambda))$ and $L(\mu)$ belong to the same block as explained in §2.) We can assume that there is a nonzero Ext^1-between the irreducible modules $L(d(\lambda))$ and $L(\mu)$.

Case 1. Suppose that $d(\lambda) \rhd \mu$, so that, by the universal mapping property of standard modules, $0 \neq [\Delta(d(\lambda)) : L(\mu)] = [P(\mu) : \Delta(d(\lambda))]$. In terms of the category $S(r,s)$-mod, this means that $[Y(\mu') : \Delta(d(\lambda)')] \neq 0$. Hence, $d(\lambda)'$ and μ' belong to the

[6]More precisely, $D : S(a,b)\text{-mod} \to S(a,b)\text{-mod}$ is a contravariant equivalence such that $D^2 \cong \mathrm{Id}$ and $DL(\lambda) \cong L(\lambda)$ for all $\lambda \in \Lambda^+(a,b)$.

same block for the weight poset $\Lambda^+(r,s)$ of $S(r,s)$-mod. Now we must show that $\mu' = d(\tau)'$ for some $\tau \in \Lambda^+(r)$. Again, we can reduce to the case where there is a nonzero Ext^1 between $L(d(\lambda)')$ and $L(\mu')$ in $S(r,s)$-mod. Using Lemma 15, we see that either $0 \neq [\Delta(d(\lambda)') : L(\mu')] = [\Delta(\lambda')^{(1)} \otimes St : L(\mu')]$ (i.e., $d(\lambda)' \rhd \mu'$) or $0 \neq [P(d(\lambda)') : \Delta(\mu')] = [P(\lambda')^{(1)} \otimes \mathrm{St} : \Delta(\mu')]$ (i.e., $\mu' \rhd d(\lambda)'$). In either case, the exactness of the functor F_1 implies that $\mu' = p\tau' + (p-1)\rho_r$ for some $\tau \in \Lambda^+(r)$, as required.

Case 2. Now suppose that $d(\lambda) \lhd \mu$. Then we can assume that $0 \neq [\Delta(\mu) : L(d(\lambda))] = [P(d(\lambda)) : \Delta(\mu)]$ in $S(s,s)$-mod$[\Lambda(r,s)']$. This means that, back in $S(r,s)$-mod, we have $[Y(d(\lambda)') : \Delta(\mu')] \neq 0$. Thus, $d(\lambda)'$ and μ' belong to the same block in $\Lambda^+(r,s)$ for $S(r,s)$-mod. Then repeating the argument as in Case 1, we conclude that $\mu = d(\tau)$ for some $\tau \in \Lambda^+(r)$.

This completes the proof. □

The highest weight category $S(s,s)$-mod$[\Lambda^+(r,s)']$ has the form A-mod for a quasi-hereditary algebra $A \cong S(s,s)/J$ for some idempotent ideal J in $S(s,s)$. Let $B(r)$ be the product of the blocks of A whose irreducible modules have the form $L(d(\lambda))$ for some $\lambda \in \Lambda^+(r)$. Thus, $B(r) = eAe$ for a central idempotent $e \in A$. The functor $\mathfrak{J} : S(r,r)$-mod $\to S(s,s)$-mod$[\Lambda(r,s)']$ induces a functor $\mathfrak{J} : S(r,r)$-mod $\to B(r)$-mod. We thank Alison Parker for pointing out an error in the original version of the following result.

Theorem 17. *The functor* $\mathfrak{J}$ *induces an equivalence* $\mathfrak{J}' : S(r,r)$-mod $\to B(r)$-mod *of categories. In particular,* $S(r,r)$ *is Morita equivalent to the algebra* $B(r)$. *The category* $B(r)$-mod *is a highest weight category with poset* $\Lambda^+(r)$ *which is isomorphic to the subposet* $d(\Lambda^+(r))$ *of* $\Lambda^+(s)$. *The equivalence* $\mathfrak{J}'$ *maps standard, costandard and simple objects indexed by* $\lambda \in \Lambda^+(r)$ *to the corresponding standard, costandard and simple objects in* $B(r)$-mod *indexed by* $d(\lambda)$.

Proof. If $P(\lambda)$ is the projective indecomposable cover of $\Delta(\lambda)$ in $S(r,r)$-mod, then $\mathfrak{J}P(\lambda) \cong P(d(\lambda))$ is the projective indecomposable cover of $\Delta(d(\lambda))$ in $B(r)$-mod, and Lemma 16 implies that all projective indecomposable $B(r)$-modules arise this way. Furthermore, using Lemma 4(d) and the definition of the functor $\mathfrak{J}$, we have

$$\mathrm{Hom}_{S(r,r)}(P(\lambda), P(\mu)) \cong \mathrm{Hom}_{B(r)}(\mathfrak{J}P(\lambda), \mathfrak{J}P(\mu))$$

for all $\lambda, \mu \in \Lambda^+(r)$. Now

$$P = \bigoplus\nolimits_{\lambda \in \Lambda^+(r)} P(\lambda), \quad \mathfrak{J}P = \bigoplus\nolimits_{\lambda \in \Lambda^+(r)} P(d(\lambda))$$

are projective generators for $S(r,r)$-mod and $B(r)$-mod, respectively. We have $C := \mathrm{End}_{S(r,r)}(P)^{\mathrm{op}}$ is isomorphic to $B := \mathrm{End}_{B(r)}(\mathfrak{J}P)^{\mathrm{op}}$, while $S(r,r)$-mod is equivalent to C-mod and $B(r)$-mod is equivalent to B-mod. Thus, $S(r,r)$-mod and $B(r)$-mod are equivalent by an equivalence $\mathfrak{J}'$ agreeing with $\mathfrak{J}$ on any object $P(\lambda)$.

Observe that, given $\lambda, \mu \in \Lambda^+(r)$, $\lambda \trianglelefteq \mu$ if and only if $d(\lambda) \trianglelefteq d(\mu)$. Also, by (19), $\mathfrak{J}'$ maps $P(\lambda)$ to $P(d(\lambda))$. Hence, $\mathfrak{J}'L(\lambda) \cong L(d(\lambda))$. It follows that $\mathfrak{J}'$ maps the injective envelope $I(\lambda)$ of $L(\lambda)$ to the injective injective envelope $I(d(\lambda))$ of $L(d(\lambda))$ in $B(r)$-mod. Since $\nabla(\lambda)$ (resp., $\nabla(d(\lambda))$) is the largest submodule of $I(\lambda)$ (resp., $I(d(\lambda))$) all of whose composition factors $L(\mu)$ satisfy $\mu \trianglelefteq \lambda$ (resp., $\mu \trianglelefteq d(\lambda)$), it follows that $\mathfrak{J}'\nabla(\lambda) \cong \nabla(d(\lambda))$. Similarly, $\mathfrak{J}'\Delta(\lambda) \cong \Delta(d(\lambda))$. □

As a consequence, we obtain the following cohomological result.

Corollary 18. (a) *Let $M, N \in S(r,r)$-mod. Regard $\mathfrak{J}'M, \mathfrak{J}'N \in S(s,s)$-mod. Then*

$$\mathrm{Ext}^\bullet_{S(r,r)}(M,N) \cong \mathrm{Ext}^\bullet_{S(s,s)}(\mathfrak{J}'M, \mathfrak{J}'N).$$

(b) *Let $W, Z \in \{L, \Delta, \nabla, Y\}$. Then for $\lambda, \mu \in \Lambda^+(r)$, we have*

$$\mathrm{Ext}^\bullet_{S(r,r)}(W(\lambda), Z(\mu)) \cong \mathrm{Ext}^\bullet_{S(s,s)}(W(d(\lambda)), Z(d(\mu))).$$

Proof. The equivalence $\mathfrak{J}'$ induces a full embedding $D^b(S(r,r)\text{-mod}) \to D^b(B(r)\text{-mod})$ of bounded derived categories. Because $B(r)$-mod is a direct product of blocks of $S(s,s)$-mod$[\Lambda^+(r,s)']$, there is another full embedding

$$D^b(B(r)\text{-mod}) \to D^b(S(s,s)\text{-mod}[\Lambda^+(r,s)'])$$

of derived categories. On the other hand, $\Lambda^+(r,s)'$ is an ideal in $\Lambda^+(s)$, so that the recollement diagram (1) provides another full embedding

$$D^b(S(s,s)\text{-mod}[\Lambda^+(r,s)']) \to D^b(S(s,s)\text{-mod}).$$

Thus, we obtain by composition a full embedding $D^b(S(r,r)\text{-mod}) \to D^b(S(s,s)\text{-mod})$. Since these full embeddings are induced by exact functors on respective module categories, assertion (a) is clear from (2) (and the accompanying footnote).

Finally, (b) follows from (a) since, by construction, $\mathfrak{J}'W(\lambda) \cong W(d(\lambda))$ for all $W \in \{L, \Delta, \nabla, Y\}$. For example, $\mathfrak{J}'Y(\lambda)$ is the indecomposable tilting module in $B(r)$-mod of highest weight $d(\lambda)$. Since it is indecomposable, it remains the indecomposable tilting module in $S(s,s)$-mod$[\Lambda^+(r,s)']$ of highest weight λ. However, the functor $i_* : S(s,s)\text{-mod}[\Lambda^+(r,s)'] \to S(s,s)$-mod takes indecomposable tilting modules to indecomposable tilting modules. Therefore, $\mathfrak{J}'Y(\lambda)$ identifies with $Y(d(\lambda))$ as an $S(s,s)$-module. □

Remark 19. Let a be a positive integer. The functor $\mathfrak{J}' : S(r,r)\text{-mod} \to S(s,s)$-mod $[\Lambda^+(r,s)']$ can be replaced by a functor

$$\mathfrak{J}'_a : S(r,r)\text{-mod} \to S(s_a, s_a)\text{-mod}[\Lambda^+(r,s_a)'],$$

where $s_a = p^a r + (p^a - 1)\frac{r(r-1)}{2}$. It is defined in the same way as $\mathfrak{J}'$, but replacing the "twisting" functor $F_1 : S(r,r)\text{-mod} \to S(r, pr + (p-1)p(p-1)/2)$ by the functor

$$F_a : S(r,r)\text{-mod} \to S(r, p^a r + (p^a-1)r(r-1)/2), \quad M \mapsto M^{(r)} \otimes \mathrm{St}_a,$$

where $M^{(r)}$ denotes the rational $\mathrm{GL}_r(k)$-module obtained by making $g \in \mathrm{GL}_r(k)$ act through $F^a(g)$, and St_a denotes the Steinberg module of highest weight $(p^a-1)\rho_r$. The straightforward modifications in the proofs are left to the reader.

6. Variations on the theme

In this section, we mainly discuss (very briefly) how some of the previous results work in a quantized situation. Throughout this section, k will be a fixed algebraically closed field of characteristic $p \geqslant 0$. Let $\zeta \in k$ be a primitive ℓth root of unity. Some further restrictions will be placed on ℓ; these will be discussed below at the appropriate place. We view k as an algebra over the ring $\mathcal{Z} := \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials, under the map $\mathcal{Z} \to k$, $v \mapsto \zeta$.

1. Quantum groups at a root of unity

Because we want to consider quantum enveloping algebras over fields of positive characteristic, we will closely follow the discussion given by Andersen-Wen [AW]. Let $A = [a_{i,j}]$ be an $n \times n$ (classical) Cartan matrix corresponding to a finite root system Φ. Fix a set $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of simple roots for Φ; hence, $a_{i,j} = \langle \alpha_i, \alpha_j^\vee \rangle$. Let X be the lattice of integral weights for Φ and let X^+ be the corresponding set of dominant weights.

Now let U be the quantum enveloping algebra over $\mathcal{Z}$ corresponding to A; U has generators $E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1}$, $1 \leqslant i \leqslant n$, $m \geqslant 0$, and forms a $\mathcal{Z}$-form of the quantum algebra over $\mathbb{Q}(v)$ with generators $E_i, F_i, K_i^{\pm 1}$ satisfying the quantum Serre relations which we do not repeat here. We let U^0 be the subalgebra of U generated by the $K_i^{\pm 1}$ and the

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-v+s-1}}{v^s - v^{-s}}$$

$c \in \mathbb{Z}$, $t \in \mathbb{Z}_{\geqslant 0}$.

Put $U_k := k \otimes_{\mathcal{Z}} U$, and let $\mathcal{C} = U_k$-mod be the category of finite-dimensional integrable type 1 U_k-modules. Then using standard results of [APW1], [APW2], [AW], in particular, the quantum version of Kempf's theorem[7], it follows that U_k-mod is a highest weight category with weight poset $\Lambda = X^+$. An argument is provided in [DS] for a somewhat weaker result—entirely adequate for our purposes here—that the modules in $\mathcal{C}$ form a "k-finite highest weight category."

Given $I \subseteq \{1, \dots, n\}$, let $X^{+(I)} \subseteq X$ be the set of weights $\lambda \in X$ satisfying $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$ for all $i \in I$. Let $U_{Ik} = k \otimes U_I$, where U_I is the "Levi" subalgebra of U generated by the $E_i^{(m)}$, $F_i^{(m)}$, for $i \in I$ and $1 \leqslant j \leqslant n$, together with U_k^0. Let $\mathcal{C}_I = U_{Ik}$-mod be the category of integrable, type 1 modules for the quantum algebra U_{Ik}. It is a highest weight category with weight poset $X^{+(I)}$. For $\lambda \in X^{+(I)}$, let $\Delta_I(\lambda)$, $\nabla_I(\lambda)$, $L_I(\lambda)$ denote the corresponding standard, costandard, and irreducible objects.

Now the main results of §3 carry through with little but notational change: Fix a coset $\Omega = \omega + \mathbb{Z}I$ of $\mathbb{Z}I$ in X. For any finite subset $F \subset \Omega^+$, *there is an equivalence of categories* $\mathcal{C}(\Omega_F^+) \cong \mathcal{C}_I[\Omega_F^+]$. In particular, if $\lambda, \mu \in \Omega^+$, then

$$\operatorname{Ext}^\bullet_{U_k}(\Delta(\lambda), L(\mu)) \cong \operatorname{Ext}^\bullet_{U_{Ik}}(\Delta_I(\lambda), L_I(\mu))$$

and

$$\operatorname{Ext}^\bullet_{U_k}(L(\mu), \nabla(\lambda)) \cong \operatorname{Ext}^\bullet_{U_{Ik}}(L_I(\mu), \nabla_I(\lambda)).$$

[7]This is proved in [APW1], [AW] with some restrictions on ℓ; however, these restrictions are completely removed in [RH].

We remark that when k has characteristic 0, the dimensions of $\mathrm{Ext}^i_{\mathcal{C}}(\Delta(\lambda), L(\mu))$ and $\mathrm{Ext}^i_{\mathcal{C}}(L(\mu), \nabla(\lambda)))$ can be computed in terms of Kazhdan–Lusztig polynomials, using the fact that, thanks to [KL] and [KT], $\mathcal{C}$ often has a Kazhdan–Lusztig theory in the sense of [CPS3]. See also [T]. The latter survey notes some restrictions on ℓ are required when Φ is not of type $\mathtt{A}$—e.g., ℓ must be unequal to 2 for type $\mathtt{D}_{2n+1}, n > 1$, and $\mathtt{E}_8$ requires ℓ be at least 25. (But the analogous restrictions in the non-simply laced cases are not known explicitly.) Strictly speaking, the results in [T] only give character formulas. However, these results are sufficient in the framework of [CPS4] to establish homological results, in particular a "Kazhdan–Lusztig theory", in the presence of suitable "pre-Hecke operators." In the quantum case, these can be constructed when $\ell \geqslant h$ (the Coxeter number) using wall-crossing functors. Now [KL] provides an equivalence (with the mentioned restrictions on ℓ) of the finite-dimensional modules in the quantum ℓth root of unity case with the full category of all finite length modules for an affine Lie algebra which have only certain composition factors (all of level $-\ell - h$), and which are direct sums of weight spaces. This makes the requisite pre-Hecke operators available for the latter category, at least when $\ell \gg h$. We conjecture they are then available for $\ell > 1$ (with other restrictions on ℓ as above, for types different from type $\mathtt{A}$) by virtue of Kumar's translation functors [Ku].

2. The special case of *q*-Schur algebras

We maintain the notation of the previous subsection, and put $q = \zeta^2$. The q-Schur algebras $S_q(n, r)$ arise in the special case of type $\mathtt{A}_{n-1}$. Explicitly, let $\widetilde{V}$ be a free $\mathcal{Z}$-module of rank r. Let $\widetilde{H}$ be the Hecke algebra over $\mathcal{Z}$ corresponding to the symmetric group $\mathfrak{S}_r$. There is a natural right action of $\widetilde{H}$ on $\widetilde{V}^{\otimes r}$ and $\widetilde{S}_q(n, r) := \mathrm{End}_{\widetilde{H}}(\widetilde{V}^{\otimes r})$ [DD, (3.1.5)]. Over k, we just set $S_q(n, r) := k \otimes \widetilde{S}_q(n, r)$. Then $S_q(n, r)$-mod is a highest weight category for weight poset $\lambda^+(n, r)$. Now all the results of §4 hold in the q-Schur algebra case. They can be proved using either the quantum algebra U_k of type $\mathtt{A}_{n-1}$ discussed above or some version of the quantum general linear group [DD], [PW]. We leave further details to the interested reader.

Finally, we leave as an interesting open question the issue of extending the results of §5 to the case of the q-Schur algebras. In relation to [Le] and this question, see [PS2, (6.6b)].

3. Other variations

Most recently, we have been attracted to results such as those in Section 3 for their potential application in the setting of characteristic p affine Lie algebras. We believe that the results do carry over, with an appropriate formalism, but prefer to leave such a development to a future project. Similarly, we believe that results very similar to those described in the present paper hold in the G_1T setting of [C].

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