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CELLS AND q-SCHUR ALGEBRAS

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Abstract. This paper shows how the Kazhdan–Lusztig theory of cells can be directly applied to establish the quasi-heredity of q-Schur algebras. The application arises because of a very strong homological property enjoyed by certain cell filtrations for q-permutation modules.

Introduction

This note points out a direct connection between the quasi-heredity of q-Schur algebras and the Kazhdan–Lusztig theory [KL1] of cells for Coxeter groups. The connection is based on a remarkable homological property – discovered in [DPS] – enjoyed by cell filtrations of certain induced (or q-permutation) modules for Hecke algebras.

More precisely, consider the generic Hecke algebra $\widetilde{H}$ over $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ associated to the symmetric group $W = \mathfrak{S}_r$. Let $\widetilde{V}$ be a free $\mathcal{Z}$-module of rank n. The Hecke algebra $\widetilde{H}$ has a natural right action on tensor space $\widetilde{V}^{\otimes r}$, and the corresponding q-Schur algebra can be defined as the endomor-

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phism algebra $\mathrm{End}_{\widetilde{H}}(\widetilde{V}^{\otimes r})$. In practice, we work with the Morita equivalent algebra

$$(1) \qquad \widetilde{S}_q(n,r) = \mathrm{End}_{\widetilde{H}}\Big(\bigoplus_{\lambda\in\Lambda^+(n,r)} \mathrm{ind}_{\widetilde{H}_\lambda}^{\widetilde{H}} \mathrm{IND}_\lambda\Big).$$

Here $\Lambda^+(n,r)$ is the set of partitions of r into at most n parts, $\widetilde{H}_\lambda$ is the parabolic subalgebra corresponding to $\lambda \in \Lambda^+(n,r)$, and IND_λ is the index character on $\widetilde{H}_\lambda$. (See Section 2 for further explanation of notation.) The algebras $\widetilde{S}_q(n,r)$, which behave well with respect to base change, go back to the work of Dipper–James [DJ2]. They derive their importance, in part, from their close relationship with the non-describing representation theory of finite general linear groups $GL(n,q)$; cf. [D].

There is a preorder $\leq_{LR}$ defined on W whose cells (equivalence classes for the relation $x \leq_{LR} y$ & $y \leq_{LR} x$) are the two-sided Kazhdan–Lusztig cells of W. Denoting this set of two-sided cells by Ξ, $\leq_{LR}$ defines a poset structure on this set. Let $\leq_{LR}^{\mathrm{op}}$ denote the *opposite* poset structure.

Main Theorem. *The algebra $\widetilde{A} = \widetilde{S}_q(n,r)$ is $\mathcal{Z}$-quasi-hereditary. In particular, for any field k and ring homomorphism $\mathcal{Z} \to k$, the algebra $\widetilde{A}_k = \widetilde{A} \otimes_{\mathcal{Z}} k$ is quasi-hereditary. There exists a subposet $(\Xi(n,r), \leq_{LR}^{\mathrm{op}})$ of $(\Xi, \leq_{LR}^{\mathrm{op}})$ which serves, for all k, as the weight poset for the associated highest weight category $\widetilde{A}_k$*–mod. *There exist $\widetilde{A}$-modules $\widetilde{\Delta}(\xi)$, $\xi \in \Xi(n,r)$, such that, for any k as above, the $\widetilde{A}_k$-modules $\widetilde{\Delta}(\xi)_k$ obtained from $\widetilde{\Delta}(\xi)$ by base-change, are the standard objects for the highest weight category $\widetilde{A}_k$*–mod.

The notion of a $\mathcal{Z}$-quasi-hereditary algebra first appeared in [CPS]. The proof that $\widetilde{A}_k$ is quasi-hereditary was first given in [PW; (11.5.2)]). In [PW] it was assumed that $q^{1/2} \in k$ and that either q is not a root of unity or $\pm q$ is an odd root of unity. Using that result, together with some (unpublished) work of Du–Scott, [CPS; (3.7.1)] established the quasi-heredity of $\widetilde{A}$ (over $\mathbb{Z}[q^{\pm 1/2}]$). In recent work, Donkin [Do; §4] has also given an argument for the quasi-heredity of $\widetilde{A}_k$ without restriction on q. His argument makes use of the representation theory of the quantum GL_n studied in [DD] (which has the advantage of avoiding the requirement in [PW] that $q^{1/2} \in k$); see also [M; p.200].

However, the cell theory point of view in the present paper is very different from that used in the arguments mentioned above. (The methods are also completely unrelated to the theory of cellular algebras [GL].) It is part of a new general approach to Hecke endomorphism algebras. Also, as stated *in its entirety*, the Main Theorem, together with its proof, proves in type A a conjecture made in [DPS; (2.5.2)] – a statement about cell modules and stratifications for very general Hecke algebras. See Corollary 2.11 and its proof for further details. The conjecture demands a very specific relationship between $\widetilde{A}$ and Kazhdan–Lusztig cell theory. (In particular, it is not enough to know just that $\widetilde{A}$ is $\mathcal{Z}$-quasi-hereditary.)

The paper is organized as follows. In Section 1, we recall from [CPS] the definition of a $\mathcal{Z}$-quasi-hereditary algebra. (Apart from this definition, we also require the local characterization of $\mathcal{Z}$-quasi-heredity given in [CPS; (3.3)].) The main result in this section, Theorem 1.7, gives a new criterion for checking the $\mathcal{Z}$-quasi-heredity of endomorphism algebras. Since the quasi-heredity property behaves well under base change, this result gives in many cases a new, and often easier, way to check that algebras over fields are quasi-hereditary. In Section 2, we first verify that the hypotheses of (1.7) hold for the algebras (1), using the dominance order on $\Lambda^+(n,r)$. The main ingredient here, apart from the homological property of cell filtrations of certain induced modules, is some representation theory of symmetric groups (already known to Frobenius). Using [KL1; (1.4)] and some elementary "equal characteristic" Brauer theory, we easily connect this theory to that for the Hecke algebra. We then prove the Main Theorem. Our proof is essentially combinatorial and elementary. Also, the integral q-Schur algebra setting results in a simpler, more general proof of quasi-heredity than the field setting provides. Finally, using deeper results (not necessary for the proof of the Main Theorem) involving perverse sheaves and Kazhdan–Lusztig polynomials [BB], [BK], and [KL2], we show in (2.13(b),(c)) that the formulation of the Main Theorem is "the same" as that in [CPS].

Notation and preliminaries

A *quasi-poset* is a pair $(\Lambda, \leq)$ consisting of a finite set Λ and a preorder (i. e., a reflexive and transitive relation) $\leq$ on it. Putting $x \sim y$ if and only if $x \leq y$ and $y \leq x$ defines an equivalence relation on Λ, whose equivalence classes are called the *cells* of Λ. In Section 2, we will use the preorders $\leq_L$ and $\leq_{LR}$ defined in [KL1] on a Coxeter group W, together with the corresponding equivalence relations $\sim_L$ and $\sim_{LR}$.

If R is a ring, then we let R-mod (resp., mod-R) denote the category of left (resp., right) finitely generated R-modules. Also, let $\mathrm{Irr}(R)$ be the set of irreducible right R-modules.

Throughout this paper, $\mathcal{Z}$ denotes a regular (commutative) ring of Krull dimension ≤ 2 and K is its field of fractions. If $\widetilde{M}$ is a $\mathcal{Z}$-module, let $\widetilde{M}_K = \widetilde{M} \otimes_{\mathcal{Z}} K$. In Section 2, we take $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$, the ring of integral Laurent polynomials in an indeterminate q. The hypotheses on $\mathcal{Z}$ will be particularly useful because of the following commutative algebra fact:

(0.1) **Lemma.** *Let $\mathcal{Z}$ be a regular (commutative) ring of Krull dimension ≤ 2. Let $\widetilde{Y}$ be a finitely generated $\mathcal{Z}$-module. Then*

(a) *The dual $\mathcal{Z}$-module $\widetilde{Y}^* = \mathrm{Hom}_{\mathcal{Z}}(\widetilde{Y}, \mathcal{Z})$ is projective.*

(b) *Assume that $\widetilde{Y}$ is a direct summand of a $\mathcal{Z}$-module of the form $\widetilde{X}^*$ for some finitely generated $\mathcal{Z}$-module $\widetilde{X}$. Let $\mathcal{V}$ be any k-subspace of $\widetilde{Y}_K$ and put $\widetilde{V} = \widetilde{Y} \bigcap \mathcal{V}$. Then $\widetilde{V}$ is a finitely generated projective $\mathcal{Z}$-module.*

Proof. Because localization at a prime ideal is an exact functor, we can assume that $\mathcal{Z}$ is a regular local ring of Krull dimension ≤ 2. Then (a) is precisely [AG; Cor., p. 17]. As for (b), observe that (a) implies that $\widetilde{Y}$ is $\mathcal{Z}$-projective. Hence, $\widetilde{Y}$ can be identified with its image in $\widetilde{Y}_K$, so that $\widetilde{V} = \widetilde{Y} \bigcap \mathcal{V}$ is defined (and is finitely generated). A linear algebra argument shows that $\widetilde{V}$ is isomorphic to its double dual $\widetilde{V}^{**}$ and hence, by (a), is projective over $\mathcal{Z}$. (The elementary details that $\widetilde{V} \cong \widetilde{V}^{**}$ are given in [DPS; (1.2.11)].) □

Section 1: A quasi-heredity criterion

Let $\widetilde{H}$ be a $\mathcal{Z}$-algebra, which is free of finite rank as a $\mathcal{Z}$-module. In Section 2, we take $\widetilde{H}$ to be a generic Hecke algebra. Let $\widetilde{T}$ be a right $\widetilde{H}$-module, which is finitely generated and projective over $\mathcal{Z}$. Suppose there is a direct sum decomposition

$$\widetilde{T} = \bigoplus_{\lambda \in \Lambda} \widetilde{T}_\lambda^{\oplus m_\lambda} \tag{1.1}$$

in which Λ is some finite indexing set. Suppose there is also a collection $\{\widetilde{S}_\lambda\}_{\lambda\in\Lambda}$ of $\widetilde{H}$-modules, each of which is finitely generated and projective over $\mathcal{Z}$. We assume, for each $\lambda \in \Lambda$, that there is fixed an increasing filtration $\widetilde{F}_\lambda$ of $\widetilde{H}$-module $\widetilde{T}_\lambda$ with sections of the form $\widetilde{S}_\nu$ for various $\nu \in \Lambda$; explicitly:

$$\begin{cases} \widetilde{F}_\lambda : 0 = \widetilde{F}_\lambda^0 \subseteq \widetilde{F}_\lambda^1 \subseteq \ldots \subseteq \widetilde{F}_\lambda^{t(\lambda)} = \widetilde{T}_\lambda, \\ \text{where } \mathrm{Gr}^i \widetilde{F}_\lambda \overset{\text{def}}{=} \widetilde{F}_\lambda^{i+1}/\widetilde{F}_\lambda^i \cong \widetilde{S}_{\nu_{\lambda,i}},\ \nu_{\lambda,i} \in \Lambda,\ 0 \leq i < t(\lambda). \end{cases} \tag{1.2}$$

The following result makes essential use of the hypotheses on the ring $\mathcal{Z}$. (See above (0.1).)

(1.3) **Lemma.** *With the above assumptions, put $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$. Assume, for $\lambda \in \Lambda$, that*

$$\mathrm{Ext}^1_{\widetilde{H}}(\widetilde{T}_\lambda/\widetilde{F}_\lambda^i, \widetilde{T}) = 0, \quad 0 \leq i < t(\lambda). \tag{1.3.1}$$

For $\lambda \in \Lambda$, form the left $\widetilde{A}$-modules

$$\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T}) \text{ and } \widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T}).$$

Then

(a) *For* $\lambda \in \Lambda$, $\widetilde{\Delta}(\lambda)$ *is a finitely generated projective* $\mathcal{Z}$*-module. Also,* $\widetilde{A}$ *is a projective* $\mathcal{Z}$*-module.*

(b) *For a* $\lambda \in \Lambda$, *form the projective* $\widetilde{A}$*-module* $\widetilde{P}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T})$. *Putting* $\widetilde{G}_i^\lambda \stackrel{\text{def}}{=} \mathrm{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda/\widetilde{F}_\lambda^i, \widetilde{T})$, $0 \le i < t(\lambda)$, *defines a (decreasing) filtration* $\widetilde{G}^\lambda$ *of the* $\widetilde{A}$*-module* $\widetilde{P}(\lambda)$ *with sections of the form* $\widetilde{\Delta}(\nu)$, *for various* $\nu \in \Lambda$; *explicitly, we have*

$$(1.3.2)\qquad \begin{cases} \widetilde{G}^\lambda : \widetilde{P}(\lambda) = \widetilde{G}_0^\lambda \supseteq \ldots \supseteq \widetilde{G}_{t(\lambda)}^\lambda = 0, \\ \text{where } \mathrm{Gr}_i\widetilde{G}^\lambda \stackrel{\text{def}}{=} \widetilde{G}_i^\lambda/\widetilde{G}_{i+1}^\lambda \cong \widetilde{\Delta}(\nu_{\lambda,i}),\ 0 \le i < t(\lambda). \end{cases}$$

(c) $\widetilde{P} \stackrel{\text{def}}{=} \bigoplus_\lambda \widetilde{P}(\lambda)$ *is a (finitely generated) projective generator for* $\widetilde{A}$*-*mod.

Proof. To prove (a), fix $\lambda \in \Lambda$. We first show that $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T})$ is a finitely generated projective $\mathcal{Z}$-module. Because $\widetilde{T}$ is $\mathcal{Z}$-projective, $\widetilde{Y} = \mathrm{Hom}_{\mathcal{Z}}(\widetilde{S}_\lambda, \widetilde{T})$ is a $\mathcal{Z}$-direct summand of a direct sum of copies of $\widetilde{S}_\lambda^*$. Thus, (0.1) is applicable with $\mathcal{V} = \mathrm{Hom}_{\widetilde{H}_K}(\widetilde{S}_{\lambda K}, \widetilde{T}_K)$. We conclude that $\widetilde{\Delta}(\lambda) = \widetilde{Y} \bigcap \mathcal{V}$ is $\mathcal{Z}$-projective, as required in the first assertion of (a). The second assertion follows in the same way, now using $\widetilde{Y} = \mathrm{End}_{\mathcal{Z}}(\widetilde{T})$ and $\mathcal{V} = \mathrm{End}_{\widetilde{H}_K}(\widetilde{T}_K)$.

It is elementary that the $\widetilde{G}_i^\lambda$ defined in (1.3.2) define a filtration on $\widetilde{P}(\lambda)$. For a given i, there is a short exact sequence

$$(1.3.3)\qquad 0 \to \widetilde{F}^{i+1}/\widetilde{F}_\lambda^i \to \widetilde{T}_\lambda/\widetilde{F}_\lambda^i \to \widetilde{T}_\lambda/\widetilde{F}_\lambda^{i+1} \to 0$$

of $\widetilde{H}$-modules. Applying the functor $\mathrm{Hom}_{\widetilde{H}}(-, \widetilde{T})$ to (1.3.3) and using the vanishing condition (1.3.1), together with the long exact sequence of Ext, yields that $\mathrm{Gr}_i\widetilde{G}^\lambda \cong \widetilde{\Delta}(\nu_{\lambda,i})$ since $\widetilde{F}_\lambda^{i+1}/\widetilde{F}_\lambda^i \cong \widetilde{S}_{\nu_{\lambda,i}}$. This proves (b).

Finally, as a left $\widetilde{A}$-module, $\widetilde{A} \cong \bigoplus_\lambda \widetilde{P}(\lambda)^{\oplus m_\lambda}$. Therefore, $\widetilde{P} = \bigoplus_\lambda \widetilde{P}(\lambda)$ is a projective generator for the category $\widetilde{A}$-mod. Thus, (c) holds. □

We wish to formulate a condition guaranteeing that $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$ is $\mathcal{Z}$-quasi-hereditary. First, recall the definition of this concept [CPS; (3.2)]. (In this definition, $\mathcal{Z}$ could be an arbitrary commutative, Noetherian ring.) Let $\widetilde{A}$ be an arbitrary $\mathcal{Z}$-algebra, finitely generated and projective as a $\mathcal{Z}$-module. An ideal $\widetilde{J}$ of $\widetilde{A}$ is a *heredity ideal* provided that

$$(1.4)\qquad \begin{cases} (1)\ \widetilde{A}/\widetilde{J} \text{ is } \mathcal{Z}\text{-projective;} \\ (2)\ \widetilde{J}^2 = \widetilde{J}; \\ (3)\ \widetilde{J} \text{ is projective as a left } \widetilde{A}\text{-module;} \\ (4)\ \widetilde{E} = \mathrm{End}_{\widetilde{A}}(\widetilde{J}) \text{ is } \mathcal{Z}\text{-semisimple.} \end{cases}$$

(Recall that a $\mathcal{Z}$-algebra $\widetilde{E}$ is $\mathcal{Z}$-semisimple provided that, for every $\mathfrak{p} \in \operatorname{Spec} \mathcal{Z}$, $\widetilde{E}(\mathfrak{p}) = \widetilde{E} \otimes_{\mathcal{Z}} \mathcal{Z}_\mathfrak{p}/\mathfrak{p}\mathcal{Z}_\mathfrak{p}$ is a semisimple algebra over the residue field $\mathcal{Z}_\mathfrak{p}/\mathfrak{p}\mathcal{Z}_\mathfrak{p}$. For further discussion and references to the literature, see [CPS; §2].) In case $\widetilde{E}$ is a split semisimple $\mathcal{Z}$-algebra in (1.4(4)) – in particular, if $\widetilde{E}$ is a direct product of matrix algebras $M_n(\mathcal{Z})$ – we call $\widetilde{J}$ a *split* heredity ideal.

The algebra $\widetilde{A}$ is *$\mathcal{Z}$-quasi-hereditary* if there exists a "defining sequence"

$$0 = \widetilde{J}_0 \subset \widetilde{J}_1 \subset \ldots \subset \widetilde{J}_t = \widetilde{A} \tag{1.5}$$

of ideals in $\widetilde{A}$ such that, for $0 < i \leq t$, $\widetilde{J}_i/\widetilde{J}_{i-1}$ is a heredity ideal in $\widetilde{A}/\widetilde{J}_{i-1}$. If each $\operatorname{End}_{\widetilde{A}/\widetilde{J}_{i-1}}(\widetilde{J}_i/\widetilde{J}_{i-1})$ is a *split* $\mathcal{Z}$-semisimple algebra, then $\widetilde{A}$ is a split $\mathcal{Z}$-quasi-hereditary algebra. As shown in [CPS; (3.3)], this notion behaves well under base change; in particular, if $\mathcal{Z} \to k$ is a homomorphism of $\mathcal{Z}$ to a field k, then the algebra $\widetilde{A}_k = \widetilde{A} \otimes_{\mathcal{Z}} k$ is quasi-hereditary.

Now we can prove

(1.6) **Lemma.** *Let $\widetilde{A}$ be a $\mathcal{Z}$-algebra which is finitely generated and projective as a $\mathcal{Z}$-module. Suppose that $\{\widetilde{\Delta}(\lambda)\}_{\lambda\in\Lambda}$ is a family of (left) $\widetilde{A}$-modules indexed by a poset Λ and that the following conditions hold:*

(1) *For $\lambda \in \Lambda$, $\widetilde{\Delta}(\lambda)$ is finitely generated and projective as a $\mathcal{Z}$-module;*

(2) *For $\lambda \in \Lambda$, there exists a finitely generated projective $\widetilde{A}$-module $\widetilde{P}(\lambda)$ which has a filtration*

$$\begin{cases} \widetilde{G}^\lambda : \widetilde{G}^\lambda_0 \supseteq \widetilde{G}^\lambda_1 \supseteq \ldots \supseteq \widetilde{G}^\lambda_{t(\lambda)} = 0, \\ \text{where } \operatorname{Gr}_i \widetilde{G}^\lambda \cong \widetilde{\Delta}(\nu_{\lambda,i}), \;\; \nu_{\lambda,i} \in \Lambda. \end{cases} \tag{1.6.1}$$

(3) *In the Grothendieck group $K_0(\widetilde{A}_K\text{-mod})$, we have*

$$[\widetilde{P}(\lambda)_K] = [\widetilde{\Delta}(\lambda)_K] + \sum_{\zeta>\lambda} m_{\zeta,\lambda}[\widetilde{\Delta}(\zeta)_K], \quad \forall \lambda \in \Lambda. \tag{1.6.2}$$

(4) *For $\lambda \in \Lambda$, $\widetilde{\Delta}(\lambda)_K$ is an absolutely irreducible $\widetilde{A}_K$-module. For $\lambda \neq \zeta \in \Lambda$, $\widetilde{\Delta}(\lambda)_K \not\cong \widetilde{\Delta}(\zeta)_K$.*

(5) *$\widetilde{P} \stackrel{\text{def}}{=} \bigoplus_\lambda \widetilde{P}(\lambda)$ is a projective generator for* $\widetilde{A}$-mod.

Then $\widetilde{A}$ is a (split) $\mathcal{Z}$-quasi-hereditary algebra. If a field k is a $\mathcal{Z}$-algebra, then $\widetilde{A}_k$-mod *is a highest weight category with weight poset Λ and with standard modules $\widetilde{\Delta}(\lambda)_k$, $\lambda \in \Lambda$.*

Proof. First, we show that $\widetilde{A}$ is $\mathcal{Z}$-quasi-hereditary. Let $\widetilde{A}' = \operatorname{End}_{\widetilde{A}}(\widetilde{P})^{\text{op}}$. Since $\widetilde{P}$ is a projective generator for $\widetilde{A}$-mod, the functor

$$F(-) = \operatorname{Hom}_{\widetilde{A}}(\widetilde{P}, -) : \widetilde{A}\text{-mod} \longrightarrow \widetilde{A}'\text{-mod}$$

is an equivalence of categories. For $\lambda \in \Lambda$, let $\widetilde{P}'(\lambda) = F(\widetilde{P}(\lambda))$ and $\widetilde{\Delta}'(\lambda) = F(\widetilde{\Delta}(\lambda))$. Then hypotheses (1)–(5) hold with any $\widetilde{\Delta}(\lambda)$ replaced by $\widetilde{\Delta}'(\lambda)$ and any $\widetilde{P}(\lambda)$ replaced by $\widetilde{P}'(\lambda)$, and it suffices to prove the lemma for $\widetilde{A}'$.

Let $\Lambda_1 \subseteq \Lambda$ be the set of maximal elements in Λ. Let $e'_1 \in \widetilde{A}'$ be an idempotent so that $\widetilde{A}'e'_1 \cong \bigoplus_{\lambda\in\Lambda_1} \widetilde{P}'(\lambda)$. By (2), (3), and (4), for $\lambda \in \Lambda_1$, we have $\widetilde{P}'(\lambda) \cong \widetilde{\Delta}'(\lambda)$. Thus, we can assume, for $\lambda \in \Lambda$, there exists an index $j(\lambda)$, such that

$$\widetilde{G}^{\lambda}_{j(\lambda)} \cong \bigoplus_{\zeta\in\Lambda_1} \widetilde{\Delta}'(\zeta)^{\oplus m_{\zeta,\lambda}},$$

while, for $i < j(\lambda)$, $\nu_{\lambda,i} \notin \Lambda_1$. Assumptions (1), (4) imply that for $\lambda \neq \zeta$ in Λ, $\mathrm{Hom}_{\widetilde{A}'}(\widetilde{\Delta}'(\lambda), \widetilde{\Delta}'(\zeta)) = 0$. Thus, the trace ideal $\widetilde{J}'_1 = \widetilde{A}'e'_1\widetilde{A}'$ satisfies $\widetilde{J}'_1 \cong \bigoplus_{\lambda\in\Lambda} \widetilde{G}^{\lambda}_{j(\lambda)}$. By (1), $\widetilde{A}'/\widetilde{J}'_1$ is $\mathcal{Z}$-projective and that $\widetilde{J}'_1$ is a projective left $\widetilde{A}'$-module. Thus, conditions (1.4(1,2,3)) hold for $\widetilde{J} = \widetilde{J}_1$.

Each $\mathrm{End}_{\widetilde{A}'}(\widetilde{\Delta}'(\lambda))$ is a finitely generated $\mathcal{Z}$-module satisfying

$$\mathrm{End}_{\widetilde{A}'}(\widetilde{\Delta}'(\lambda))_K \cong K.$$

Since $\mathcal{Z}$ is integrally closed in K, $\mathrm{End}_{\widetilde{A}'}(\widetilde{\Delta}(\lambda)) \cong \mathcal{Z}$. Therefore, $\widetilde{E}_1 = \mathrm{End}_{\widetilde{A}'}(\widetilde{J}'_1)$ is a direct product of matrix algebras $M_n(\mathcal{Z})$, and hence is split $\mathcal{Z}$-semisimple. Thus, (1.4(5)) holds, and $\widetilde{J}'_1$ is a split heredity ideal of $\widetilde{A}'$.

Now the family $\{\widetilde{\Delta}'(\lambda)\}_{\lambda\in\Lambda\setminus\Lambda_1}$ satisfies our hypotheses for the algebra $\widetilde{A}'/\widetilde{J}'_1$. Inductively, it follows that $\widetilde{A}'$ is a split $\mathcal{Z}$-quasi-hereditary algebra.

To prove the last assertion, first let $k = k(\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}\ \mathcal{Z}$, where $k(\mathfrak{p}) = \mathcal{Z}_{\mathfrak{p}}/\mathfrak{p}\mathcal{Z}_{\mathfrak{p}}$. By [CPS; (3.3)], $0 = \widetilde{J}_{0k} \subset \widetilde{J}_{1k} \subset \ldots \subset \widetilde{J}_{nk}$ is a defining sequence for the quasi-hereditary algebra $\widetilde{A}_k$. For any i, $\widetilde{\Delta}(\lambda)_k$ is a projective $\widetilde{A}_k/\widetilde{J}_{ik}$-module, which is indecomposable since $\mathrm{End}_{\widetilde{A}}(\widetilde{\Delta}(\lambda)) \cong \mathcal{Z}$. If $L_k(\lambda)$ is the irreducible head of $\widetilde{\Delta}(\lambda)_k$, then it follows that the $L_k(\lambda)$, $\lambda \in \Lambda$, are the distinct (absolutely) irreducible $\widetilde{A}_k$-modules. It is now clear that $\widetilde{A}_k$-mod is a highest weight category with poset Λ and with standard objects the $\widetilde{\Delta}(\lambda)_k$, $\lambda \in \Lambda$, and that this remains true for any extension field of k. □.

Putting everything together yields the criterion below for checking that an endomorphism algebra $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$ is $\mathcal{Z}$-quasi-hereditary. In the next section, where $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$, we will see the advantage of our *integral* setup: First, elementary Brauer theory and Frobenius' representation theory for $\mathfrak{S}_r$ permit an easy verification of the conditions (1)–(3) below over $\mathcal{Z}$. Second, the Ext^1-vanishing condition (4) – which one would not expect to hold in general over a field k – does hold over $\mathcal{Z}$, thanks to the nature of the Kazhdan–Lusztig basis for $\widetilde{H}$. This is precisely the homological property mentioned in the introduction and discovered in [DPS]; see the discussion above (2.7) below.

(1.7) Theorem. *Let $\widetilde{A} = \mathrm{End}_{\widetilde{H}}(\widetilde{T})$, with $\widetilde{T}$ as in (1.1). Let $\{\widetilde{S}_\lambda\}_{\lambda\in\Lambda}$ be a collection of of $\widetilde{H}$-modules, each of which is finitely generated and projective over $\mathcal{Z}$. Suppose that each $\widetilde{T}_\lambda$ has a filtration $\widetilde{F}_\lambda$ as in (1.2). For $\lambda \in \Lambda$, put $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T}) \in \mathrm{Ob}(\widetilde{A}\text{-mod})$. In addition, assume that Λ is a poset, and that the following conditions hold:*

(1) *$\widetilde{H}_K$ is a split semisimple algebra.*

(2) *For $\lambda \in \Lambda$, $\widetilde{S}_{\lambda K}$ is an irreducible $\widetilde{H}_K$-module. If $\lambda \neq \zeta$, then $\widetilde{S}_{\lambda K} \not\cong \widetilde{S}_{\zeta K}$.*

(3) *For $\lambda \in \Lambda$, we have $\widetilde{T}_{\lambda K} = \widetilde{S}_{\lambda K} \oplus \bigoplus_{\zeta>\lambda} \widetilde{S}_{\zeta K}^{\oplus d_\zeta}$.*

(4) *For $\lambda \in \Lambda$, the Ext^1-vanishing condition (1.3.1) holds.*

Then $\widetilde{A}$ is a (split) $\mathcal{Z}$-quasi-hereditary algebra. If $\mathcal{Z} \to k$ is a homomorphism of $\mathcal{Z}$ to a field k, then the highest weight category $\widetilde{A}_k$-mod can be taken to have weight poset Λ and standard modules $\widetilde{\Delta}(\lambda)_k$, $\lambda \in \Lambda$.

Proof. We need only check the hypotheses of (1.6) hold. By (1.3), both (1.6(1)) and (1.6(2)) hold. By the definition of $\widetilde{A}$, (1.6(5)) is automatic. Finally, (1.7(3)) implies condition (1.6(3)), while (1.7(2)) yields (1.6(4)). □

Section 2: The main result

Fix a positive integer r, and let $W = \mathfrak{S}_r$ be the symmetric group of degree r. Let S be the set of involutions $(i, i+1)$, $1 \leq i < r$, so that (W, S) is a Coxeter system. Let $\ell : W \to \mathbb{Z}^+$ be the corresponding length function and let $<$ be the Chevalley-Bruhat partial order on W. Finally, recall that $\mathcal{Z} = \mathbb{Z}[q, q^{-1}]$ (q an indeterminate) in this section.

We will make some limited use of the classical characteristic zero representation theory of W. Let $\Lambda^+(r)$ be the set of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of r. For a positive integer n, let $\Lambda^+(n, r)$ be the set of partitions of r into at most n nonzero parts. Given $\lambda, \mu \in \Lambda^+(r)$, put $\lambda \trianglelefteq \mu$ if $\lambda_1 + \ldots + \lambda_i \leq \mu_1 + \ldots + \mu_i$ for all i. Then $\trianglelefteq$ defines a poset structure (the dominance order) on $\Lambda^+(r)$, and $\Lambda^+(n, r)$ is a coideal for all n. For $\lambda \in \Lambda^+(r)$, let $\mathcal{Y}(\lambda)$ be the Young diagram of shape λ. Let $\mathbf{t}^\lambda$ be the tableau of shape λ obtained by filling in the boxes in the first row of $\mathcal{Y}(\lambda)$ consecutively with the integers $1, 2, \dots, \lambda_1$, the second row with the integers $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$, etc. The group W acts on the set of all tableaux of shape λ with entries $1, 2, \dots, r$ and the Young (or parabolic) subgroup W_λ is defined to be the row stabilizer of $\mathbf{t}^\lambda$. Thus, $W_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \dots$.

The generic Hecke algebra $\widetilde{H}$ is a $\mathcal{Z}$-algebra which is a free $\mathcal{Z}$-module with basis $\{\tau_w\}_{w\in W}$, satisfying the relations

$$(2.1) \qquad \tau_s \tau_w = \begin{cases} \tau_{sw}, & sw > w, \\ q\tau_{sw} + (q-1)\tau_w, & sw < w \end{cases} \qquad (s \in S, w \in W).$$

We form the discrete valuation ring $\mathcal{O} = \mathbb{Q}[q^{\pm 1}]_{(q-1)}$. If $\mathfrak{m}$ denotes its unique maximal ideal, then $\mathcal{O}/\mathfrak{m} \cong \mathbb{Q}$, while $K = \mathbb{Q}(q)$ is the fraction field of $\mathcal{O}$. The $\mathcal{O}$-algebra $\widetilde{H}' = \widetilde{H}_{\mathcal{O}}$ serves as an intermediary between the algebras $\widetilde{H}_K$ and $\widetilde{H}_{\mathcal{O}/\mathfrak{m}} = \widetilde{H}_{\mathbb{Q}} \cong \mathbb{Q}W$. The group algebra $\mathbb{Q}W$ is classically a split semisimple algebra, as is the Hecke algebra $\widetilde{H}_K$ by [BC]. For $\mathcal{L} \in \operatorname{Irr}(\widetilde{H}_K)$, choose an $\widetilde{H}'$-lattice $\widetilde{L}$ for $\mathcal{L}$, i. e., an $\widetilde{H}'$-module $\widetilde{L}$ which is free over $\mathcal{O}$ and satisfies $\widetilde{L}_K \cong \mathcal{L}$. Let $\widetilde{L}_{\mathbb{Q}} = \widetilde{L} \otimes_{\mathcal{O}} \mathbb{Q} \in \operatorname{Ob}(\text{mod-}\mathbb{Q}W)$. The elementary theory of Cartan matrices for finite rank algebras over discrete valuation rings is now applicable; in particular, [B; (1.9.6)] implies that the $\mathcal{L} \mapsto \widetilde{L}_{\mathbb{Q}}$ is a well-defined (dimension preserving) bijection $\operatorname{Irr}(\widetilde{H}_K) \to \operatorname{Irr}(\mathbb{Q}W)$ from irreducible $\widetilde{H}_K$-modules to irreducible $\mathbb{Q}W$-modules. Classically, $\operatorname{Irr}(\mathbb{Q}W)$ is indexed by $\Lambda^+(r)$. For $\lambda \in \Lambda^+(r)$, let S_λ be the associated irreducible $\mathbb{Q}W$-module, and let $\widetilde{S}_\lambda^K \in \operatorname{Irr}(\widetilde{H}_K)$ correspond to S_λ under the above bijection.

We need to make use of some elementary Kazhdan–Lusztig cell theory for the Coxeter group W, [KL1]. For $(x, y) \in W \times W$, let $P_{x,y}$ be the associated Kazhdan–Lusztig polynomial, and set $C'_y = q^{-\ell(y)/2} \sum_{x \leq y} P_{x,y} \tau_x \in \widetilde{H}_0 = \widetilde{H} \otimes_{\mathcal{Z}} \mathbb{Z}[q^{1/2}, q^{-1/2}]$. Thus, $\{C'_y\}_{y \in W}$ is one of the two Kazhdan–Lusztig bases (considered in [KL1]) for the generic Hecke algebra $\widetilde{H}_0$ over the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. Put $C_y^+ = q^{\ell(y)/2} C'_y$, so that $\{C_y^+\}_{y \in W}$ is a $\mathcal{Z}$-basis for $\widetilde{H}$. The preorder $\leq_L$ defined on W in [KL1] has the property $hC_y^+ \in \sum_{w \leq_L y} \mathcal{Z} C_w^+$ for any $h \in \widetilde{H}$ and $y \in W$. Similarly, there is a preorder $\leq_R$ on W (given by $x \leq_R y$ if and only if $x^{-1} \leq_L y^{-1}$) and $C_y^+ h \in \sum_{w \leq_R y} \mathcal{Z} C_w^+$ for any $h \in \widetilde{H}$ and $y \in W$. If $s \in S$ and $sy < y$ (resp., $ys < y$), then $\tau_s C_y^+ = q C_y^+$ (resp., $C_y^+ \tau_s = q C_y^+$).

The cells associated to $\leq_L$ (resp., $\leq_R$) are called Kazhdan–Lusztig *left cells* (resp., *right cells*) of W. For example, the left cells are the equivalence classes in W with respect to the equivalence relation $\sim_L$ given by: $x \sim_L y \iff x \leq_L y \& y \leq_L x$. Clearly, each Kazhdan–Lusztig left cell $\omega \subseteq W$ defines a left $\widetilde{H}$-module $\widetilde{E}_\omega = \widetilde{H}^{\leq_L \omega} / \widetilde{H}^{<_L \omega}$ (i.e., a *left cell module*), where, fixing some $w \in \omega$, we put

$$\text{(2.2)} \qquad \widetilde{H}^{\leq_L \omega} = \sum_{\substack{y \in W \\ y \leq_L w}} \mathcal{Z} C_y^+ \quad \text{and} \quad \widetilde{H}^{<_L \omega} = \sum_{\substack{y \in W \\ y \leq_L w, y \not\sim_L w}} \mathcal{Z} C_y^+ .$$

A linear ordering on the set of left cells compatable with $\leq_L$ determines a filtration of $\widetilde{H}$ as a left $\widetilde{H}$-module with sections left cell modules $\widetilde{E}_\omega$.

Let $\leq_{LR}$ be the preorder on W generated by the preorders $\leq_L$ and $\leq_R$. The set Ξ of cells defined by $\leq_{LR}$ is the set of Kazhdan–Lusztig *two-sided cells* in W. The preorder $\leq_{LR}$ induces a poset structure on Ξ, still denoted $\leq_{LR}$.

For the following result, we require the $*$-operations on W. These are defined in [KL1; (4.1)], which contains all the properties we need. Given

$w \in W$, the definition of w^* depends on an explicit choice of $s, t \in S$ such that st has order 3 and such that the set $\mathcal{R}(w) \bigcap \{s,t\}$ has cardinality 1. Here $\mathcal{R}(w) = \{s \in S \,|\, ws < w\}$ is the right-set of w. Suppose that such a pair s, t has been fixed relative to $w \in W$, so that w^* is defined. If $y \sim_L w$, then $\mathcal{R}(y) = \mathcal{R}(w)$ and y^* is defined for the same pair s, t; therefore, if ω is the left cell in W containing w, we can define $\omega^* = \{y^* \,|\, y \in \omega\}$. For our purposes it will not be necessary to explicitly mention s, t when defining w^* or ω^*.

(2.3) **Lemma.** *For left cells ω, ω', the left cell modules $\widetilde{E}_\omega$ and $\widetilde{E}_{\omega'}$ are isomorphic if and only if ω and ω' are contained the same two-sided cell ξ. For $\xi \in \Xi$, let $\widetilde{E}_\xi$ denote the module $\widetilde{E}_\omega$ for any left cell ω contained in ξ. The $\widetilde{E}_{\xi K}$, $\xi \in \Xi$, are the distinct irreducible $\widetilde{H}_K$-modules.*

Proof. By [KL1; p. 177], if ω is a left cell contained in ξ, then $\omega^* = \{w^* \,|\, w \in \omega\}$ is also a left cell contained in ξ. In the generic Hecke algebra $\widetilde{H}_0$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, if $hC'_w \equiv \sum_{x \sim_L w} \alpha'_x(h,w) C'_x \mod \sum_{z <_L w} \mathcal{Z} C'_z$, then [KL1; (4.2)] proves, in effect, that $\alpha'_x(h,w) = \alpha'_{x^*}(h, w^*)$ (i. e., the left cells ω and ω^* have the same W-graph, for the C'_w-basis). If $hC^+_w \equiv \sum_{x \sim_L w} \alpha_x(h,w) C^+_x \mod \sum_{z <_L w} \mathcal{Z} C^+_z$, the definition of the C^+_w, implies that

$$\begin{aligned} \alpha_x(h,w) &= q_w^{1/2} q_x^{-1/2} \alpha'_x(h,w) = q_w^{1/2} q_x^{-1/2} \alpha'_{x^*}(h,w^*) \\ &= q_w^{1/2} q_x^{-1/2} q_{w^*}^{-1/2} q_{x^*}^{1/2} \alpha_{x^*}(h,w^*). \end{aligned}$$

Here $q_w = q^{\ell(w)/2}$, etc. Now a direct calculation verifies that the map

$$C^+_w + \sum_{z <_L w} \mathcal{Z} C^+_z \mapsto q_w^{1/2} q_{w^*}^{-1/2} q^{1/2} C^+_{w^*} + \sum_{z <_L w^*} \mathcal{Z} C^+_z, \ w \in \omega,$$

defines an isomorphism $\widetilde{E}_\omega \xrightarrow{\sim} \widetilde{E}_{\omega^*}$. (In checking this fact, it is necessary to observe that w and w^* have opposite parity.) Since any left cell in ξ can be obtained from a fixed left cell by applying a sequence of $*$-operations (by [KL1; p. 177] again), it follows that $\widetilde{E}_\omega \cong \widetilde{E}_{\omega'}$ if ω and ω' lie in the same two-sided cell ξ. Since any irreducible $\widetilde{H}_K$-module must appear as a summand of some $\widetilde{E}_{\xi K}$, the remaining statements follow by an elementary counting argument, using [KL1; (1.4)]. □

(2.4) *Remark.* Left cell modules are defined using a different Kazhdan–Lusztig basis $\{C_y\}$ of $\widetilde{H}$ in [KL1]. In [KL1; (1.4)], a version of the above lemma is established for these modules. Graham [G; (2.12)] has given an elementary combinatorial proof of this fact.

For $\xi \in \Xi$, form the dual module $\widetilde{S}_\xi = \mathrm{Hom}_{\mathcal{Z}}(\widetilde{E}_\xi, \mathcal{Z}) \in \mathrm{Ob}(\text{mod-}\widetilde{H})$. (The $\widetilde{S}_\xi$ are the *dual left cell modules* discussed in [DPS; (2.5)]. It is not hard to see, using a comparison over $\widetilde{A}_K$ and a purity argument, that $\widetilde{S}_\xi$

identifies with a Specht module for $\widetilde{H}$ as defined in [DJ1; §4]. However, we do not need this fact.) By (2.3), the $\widetilde{S}_{\xi K}$, $\xi \in \Xi$, are precisely the distinct irreducible right $\widetilde{H}_K$-modules. Thus, there is a bijection $\Xi \xrightarrow{\alpha} \Lambda^+(r)$ of *sets* satisfying $\widetilde{S}_{\xi K} \cong \widetilde{S}^K_{\alpha(\xi)}$. For $\lambda \in \Lambda^+(r)$, we sometimes write $\widetilde{S}_\lambda$ for the right $\widetilde{H}$-module $\widetilde{S}_\xi$ if $\alpha(\xi) = \lambda$.

We wish to apply this theory to q-permutation modules. If $\lambda \in \Lambda^+(r)$, let $x_\lambda = \sum_{w \in W_\lambda} \tau_w$. We will consider both the left q-permutation module $\widetilde{H}x_\lambda$ and the right q-permutation module $x_\lambda \widetilde{H}$. For convenience, write $\widetilde{T}_\lambda = x_\lambda \widetilde{H}$. The map IND : $\widetilde{H} \to \mathcal{Z}$, $\tau_w \mapsto q^{\ell(w)}$, defines a linear character, the *trivial* or *index* character, on $\widetilde{H}$. If $\widetilde{H}_\lambda$ is the subalgebra of $\widetilde{H}$ generated by all τ_w, $w \in W_\lambda$, and $\mathrm{IND}_\lambda = \mathrm{IND}|_{\widetilde{H}_\lambda}$, then we can reinterpret $\widetilde{T}_\lambda$ as the induced module $\widetilde{T}_\lambda = \mathrm{ind}^{\widetilde{H}}_{\widetilde{H}_\lambda} \mathrm{IND}_\lambda$. Because $\widetilde{H}$ admits a symmetric, associate pairing, the modules $\widetilde{H}x_\lambda$ and $x_\lambda \widetilde{H}$ are related to each other by duality:

$$\mathrm{Hom}_{\mathcal{Z}}(\widetilde{H}x_\lambda, \mathcal{Z}) \cong x_\lambda \widetilde{H}. \tag{2.5}$$

See [DJ1; §2] or [DPS; (2.1.9)].

For $\lambda \in \Lambda^+(r)$, let $W^\lambda = \{y \in W \mid ys < y, \forall s \in S \bigcap W_\lambda\}$; thus, W^λ is a set of left coset representatives of the Young subgroup W_λ in W. By [DPS; (2.3.5)], $\widetilde{H}x_\lambda$ has basis $\{C^+_y \mid y \in W^\lambda\}$. This fact, together the discussion after (2.2) above, implies that there exists a $\widetilde{H}$-module filtration of $\widetilde{H}x_\lambda$ with sections which are isomorphic to left cell modules $\widetilde{E}_\xi$. Applying the exact duality functor $(-)^* = \mathrm{Hom}_{\mathcal{Z}}(-, \mathcal{Z})$ and using (2.5), this filtration determines a filtration $\widetilde{F}_\lambda$ on $\widetilde{T}_\lambda$ with sections of the form $\widetilde{S}_\xi$, $\xi \in \Xi$. We call $\widetilde{F}_\lambda$ a *dual left cell filtration* of $\widetilde{T}_\lambda$.

For any $\lambda \in \Lambda^+(r)$, $T_\lambda \stackrel{\mathrm{def}}{=} \widetilde{T}_{\lambda\mathbb{Q}} \cong \mathrm{ind}^W_{W_\lambda} \mathbb{Q}$ is the permutation module of W_λ in W. Frobenius tells us that there is a triangular decomposition $T_\lambda \cong S_\lambda \oplus \bigoplus_{\mu \rhd \lambda} S_\mu^{\oplus d_\mu}$. Thus, for $\xi \in \Xi$,

$$\widetilde{T}_{\alpha(\xi)K} = \widetilde{S}_{\xi K} \oplus \bigoplus_{\alpha(\zeta) \rhd \alpha(\xi)} \widetilde{S}^{\oplus d_\zeta}_{\zeta K}. \tag{2.6}$$

Thus, if $\lambda \in \Lambda^+(n,r)$, then $\widetilde{S}_\mu$ appears as a section in a dual left cell filtration $\widetilde{F}_\lambda$ of $\widetilde{T}_\lambda$ only if $\mu \in \Lambda^+(n,r)$. (Keep in mind that $\Lambda^+(n,r)$ is a coideal in $\Lambda^+(r)$ with respect to the dominance order.) The filtration $\widetilde{F}_\lambda$ has the strong homological property that

$$\mathrm{Ext}^1_{\widetilde{H}}(\widetilde{T}_\lambda / \widetilde{F}^i_\lambda, \widetilde{T}_\mu) = 0, \quad \forall i,\ \mu \in \Lambda^+(r). \tag{2.7}$$

This fact is proved in [DPS; (2.3.9.2)]; by means of dimension shifting, the argument for it ultimately comes down to the fact that the Kazhdan–Lusztig basis vectors C_y^+ for the induced modules $\widetilde{H}x_\lambda$ behave well under base-change. With these preliminaries, we now prove

(2.8) **Theorem.** *The algebra $\widetilde{A} = \widetilde{S}_q(n,r)$ defined by display (1) in the introduction is a $\mathcal{Z}$-quasi-hereditary algebra. Let*

$$\widetilde{T} = \bigoplus_{\lambda\in\Lambda^+(n,r)} \widetilde{T}_\lambda. \tag{2.8.1}$$

For $\lambda \in \Lambda^+(n,r)$, define $\widetilde{\Delta}(\lambda) = \mathrm{Hom}_{\widetilde{H}}(\widetilde{S}_\lambda, \widetilde{T})$. Then for any field k which is a $\mathcal{Z}$-algebra, $\widetilde{A}_k$-mod is a highest weight category with poset $(\Lambda^+(n,r), \trianglelefteq)$ and standard objects $\widetilde{\Delta}(\lambda)_k$, $\lambda \in \Lambda^+(n,r)$.

Proof. We will apply (1.7) for the Hecke algebra $\widetilde{H}$ acting on $\widetilde{T}$ as defined in (2.8.1) with $\Lambda^+(n,r)$ given the dominance order $\trianglelefteq$. By definition, each dual left cell module $\widetilde{S}_\lambda$, $\lambda \in \Lambda^+(n,r)$, is a finitely generated free $\mathcal{Z}$-module. As discussed above, each $\widetilde{T}_\lambda$, $\lambda \in \Lambda^+(n,r)$, has a filtration $\widetilde{F}_\lambda$ as in (1.2). Conditions (1.7(1),(2)) all hold by previous discussion, while (1.7(3)) is implied by (2.6). Finally, (1.7(4)) holds by (2.7). □

We are now ready to give the *proof of the Main Theorem* from the introduction. By (2.8), $\widetilde{A}$ is $\mathcal{Z}$-quasi-hereditary. Let $\Xi(n,r) = \alpha^{-1}(\Lambda^+(n,r))$, given poset structure induced from $(\Xi, \leq_{LR}^{\mathrm{op}})$. Let $\widetilde{A}$, $\widetilde{T}$, etc. be as in (2.8). Let k be any field which is a $\mathcal{Z}$-algebra. We show that $\widetilde{A}_k$ is a highest weight category with poset $(\Xi(n,r), \leq_{LR}^{\mathrm{op}})$ and standard modules $\{\widetilde{\Delta}(\alpha(\xi))_k\}_{\xi\in\Xi(n,r)}$.

Fix $\xi \in \Xi(n,r)$. Let $\widetilde{E}$ be the $\mathcal{Z}$-submodule of $\widetilde{H}x_{\alpha(\xi)}$ spanned by all Kazhdan–Lusztig basis elements C_w^+ such that there exists $y \in W$ satisfying $w \leq_L y \in \xi$. Observe that $\widetilde{E}$ is an $\widetilde{H}$-submodule of $\widetilde{H}x_{\alpha(\xi)}$ which is evidently filtered by certain left cell modules $\widetilde{E}_\omega$. By [DPS; (2.3.7)], we can choose a filtration $\widetilde{E}^{\alpha(\xi)}$ of $\widetilde{H}x_{\alpha(\xi)}$ by left cell modules so that $\widetilde{E}_i^{\alpha(\xi)} = \widetilde{E}$ for some i. Thus, the filtration $\widetilde{G}^{\alpha(\xi)} = \mathrm{Hom}_{\widetilde{H}}(\widetilde{E}^{\alpha(\xi)*}, \widetilde{T})$ of $\widetilde{P}(\alpha(\xi))$ satisfies $\widetilde{G}_i^{\alpha(\xi)} = \widetilde{E}^{*\diamond} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\widetilde{H}}(\widetilde{E}^*, \widetilde{T})$ for some i. Also, the top filtration term $\mathrm{Gr}_i\widetilde{G}^{\alpha(\xi)}$ of $\widetilde{E}^{*\diamond}$ is $\widetilde{\Delta}(\alpha(\xi))$.

Let $P_k(\alpha(\xi))$ be the projective cover of $\widetilde{\Delta}(\alpha(\xi))_k$. Since $\widetilde{P}(\alpha(\xi))_k$ is also projective and has $\widetilde{\Delta}(\alpha(\xi))_k$ as a homomorphic image (by (2.6), (2.8) and the axioms for a highest weight category), there is a surjection $\widetilde{P}(\alpha(\xi))_k \twoheadrightarrow P_k(\alpha(\xi))$. Let $L_k(\alpha(\xi))$ denote the irreducible head of $P_k(\alpha(\xi))$. By (2.6), any $\widetilde{\Delta}$-section of $\widetilde{P}(\alpha(\xi))/\widetilde{E}^{*\diamond}$ has the form $\widetilde{\Delta}(\mu)$ for some $\mu \triangleright \alpha(\xi)$. For such a μ, (2.8) implies that $\mathrm{Hom}_{\widetilde{A}_k}(\widetilde{\Delta}(\mu)_k, L_k(\alpha(\xi))) = 0$, so that the composite $\widetilde{E}_k^{*\diamond} \to \widetilde{P}(\alpha(\xi))_k \to P_k(\alpha(\xi)) \to L_k(\alpha(\xi))$ is nonzero, and, hence, surjective. Thus, there is a surjection $\widetilde{E}_k^{*\diamond} \twoheadrightarrow P_k(\alpha(\xi))$. Therefore, $P_k(\alpha(\xi))$ is a direct summand of $\widetilde{E}_k^{*\diamond}$, and all the terms of its $\widetilde{\Delta}_k$-filtration appear among those of the $\widetilde{\Delta}_k$-filtration of $\widetilde{E}_k^{*\diamond}$. Thus, if $\widetilde{\Delta}(\alpha(\zeta))_k$ is a section in the $\widetilde{\Delta}_k$-filtration of $P_k(\alpha(\xi))$, then $\zeta \geq_{LR}^{\rm op} \xi$. Hence, $\mathrm{Ker}(P_k(\alpha(\xi)) \twoheadrightarrow \widetilde{\Delta}(\alpha(\xi))_k)$ has $\widetilde{\Delta}_k$-sections $\widetilde{\Delta}(\alpha(\zeta))_k$, $\zeta >_{LR}^{\rm op} \xi$.

Finally, if $[\widetilde{\Delta}(\alpha(\zeta))_k : L_k(\alpha(\xi))] \neq 0$ for $\zeta \neq \xi$, then

$$\mathrm{Hom}_{\widetilde{A}_k}(P_k(\alpha(\xi)), \widetilde{\Delta}(\alpha(\zeta))_k) \neq 0.$$

Let $\widetilde{\nabla}(\alpha(\xi))$ be the $\mathcal{Z}$-dual of the analogue of $\widetilde{\Delta}(\alpha(\xi))$ in the category mod–$\widetilde{A}$ of right $\widetilde{A}$-modules. Then $\widetilde{\nabla}(\alpha(\xi))_K \cong \widetilde{\Delta}(\alpha(\xi))_K$. Thus, $\widetilde{\nabla}(\alpha(\xi))$ is also a lattice for $\widetilde{\Delta}(\alpha(\xi))_K$, and a standard argument (see, e.g., [DPS; (1.1.2)]) implies that $\widetilde{\nabla}(\alpha(\xi))_k$ has the same composition factors as $\widetilde{\Delta}(\alpha(\xi))_k$. Hence,

$$\mathrm{Hom}_{\widetilde{A}_k}(P_k(\alpha(\xi)), \widetilde{\nabla}(\alpha(\zeta))_k) \neq 0.$$

Therefore, $\widetilde{\Delta}(\alpha(\zeta))_k$ appears in the Δ-filtration of $P_k(\alpha(\xi))$ by Brauer–Humphreys reciprocity. The previous paragraph implies that $\xi <_{LR}^{\rm op} \zeta$.

Thus, $\widetilde{A}_k$-mod is a highest weight category with poset $(\Xi, \leq_{LR}^{\rm op})$ and with standard objects $\{\widetilde{\Delta}(\alpha(\xi)_k\}_{\xi\in\Xi(n,r)}$. Putting $\widetilde{\Delta}(\xi) = \widetilde{\Delta}(\alpha(\xi))$, $\xi \in \Xi(n,r)$, this completes the proof of the Main Theorem.

Before proving the next result, we require the following elementary commutative algebra result:

(2.9) Lemma. *Let Z be a regular commutative ring of Krull dimension at most 2, and let B be a Z-algebra, finitely generated and projective as a Z-module. Suppose X, Y are B-modules which are finitely generated and projective over Z. Assume $\mathrm{Ext}_B^1(X,Y) \neq 0$. Then there exists $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}_B^1(X,Y))$ such that $\mathrm{Ext}_{B_k}^1(X_k, Y_k) \neq 0$, where k is the residue field $k(\mathfrak{p}) = Z_\mathfrak{p}/\mathfrak{p}Z_\mathfrak{p}$.*

Proof. The assertion is clear if $0 = \mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}_B^1(X,Y))$. This is because $\mathrm{Ext}_{B_k}^1(X_k, Y_k)$ identifies in this case with the localization $\mathrm{Ext}_B^1(X,Y)_\mathfrak{p}$. If $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}_B^1(X,Y))$ has height 1, then the localization $Z_\mathfrak{p}$ is a discrete valuation ring, in which case the result is well-known (and is an easy exercise). Thus, we may assume Krull dimension of Z is 2, and that there is a maximal ideal $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Ext}_B^1(X,Y))$, and no smaller prime ideals. Localizing at $\mathfrak{p}$, we can assume that Z is local.

The ideal $\mathfrak{p}$ must annihilate some nonzero $\epsilon \in \mathrm{Ext}_B^1(X,Y)$. Since Z is regular of Krull dimension 2, we can choose $p \in \mathfrak{p}$ so that Z/pZ is a discrete valuation ring. We claim that $\mathrm{Ext}_{B/pB}^1(X/pX, Y/pY) \cong \mathrm{Ext}_B^1(X, Y/pY) \neq 0$. The result will then follow from the discrete valuation ring case.

The natural map $\mathrm{Ext}_B^1(X,Y) \to \mathrm{Ext}_B^1(X, Y/pY)$ is nonzero, since otherwise multiplication by p is surjective as an endomorphism of $\mathrm{Ext}_B^1(X,Y)$.

But surjectivity would imply that multiplication by p is an isomorphism, whereas $p\epsilon = 0$. Thus, $\operatorname{Ext}^1_B(X, Y/pY) \neq 0$, as required. □

(2.10) **Corollary.** *Let $\widetilde{A}$, etc. be as in* (2.8). *Put*

$$\Xi(n,r) = \alpha^{-1}(\Lambda^+(n,r)).$$

We have

(a) $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{\Delta}(\alpha(\zeta)), \widetilde{\Delta}(\alpha(\xi)) = 0$ *unless* $\zeta \leq^{\mathrm{op}}_{LR} \xi$ $(\xi, \zeta \in \Xi(n,r))$.

(b) *A sequence of defining ideals* $0 = \widetilde{J}_0 \subset \widetilde{J}_1 \subset \ldots \subset \widetilde{J}_n = \widetilde{A}$ *can be chosen for the $\mathcal{Z}$-quasi-hereditary algebra $\widetilde{A}$ with respect to any listing $\xi_1, \ldots, \xi_n$ of $\Xi(n,r)$ with $i < j$ whenever $\xi_i >^{\mathrm{op}}_{LR} \xi_j$, and which satisfies* $(\widetilde{J}_i/\widetilde{J}_{i-1})_K \cong \widetilde{\Delta}(\alpha(\xi))^{\oplus d_i}_K$ *for some positive integer d_i.*

(c) *The modules* $\{\widetilde{\Delta}(\alpha(\xi))\}_{\xi \in \Xi(n,r)}$ *form a strict stratifying system for $\widetilde{A}$-*mod, *in the sense of* [DPS; (1.2.4)], *with respect to the poset* $(\Xi(n,r), \leq^{\mathrm{op}}_{LR})$.

Proof. Part (b) follows easily from (a) and the Main Theorem, after "rearranging" the filtrations of the $\widetilde{P}(\lambda)$'s. Similarly, (c) follows from (a) by taking $\widetilde{P}(\lambda) = \operatorname{Hom}_{\widetilde{H}}(\widetilde{T}_\lambda, \widetilde{T})$.

To prove (a), suppose that $\operatorname{Ext}^1_{\widetilde{A}}(\widetilde{\Delta}(\alpha(\zeta)), \widetilde{\Delta}(\alpha(\xi)) \neq 0$. By (2.9),

$$\operatorname{Ext}^1_{\widetilde{A}_k}(\widetilde{\Delta}(\alpha(\zeta))_k, \widetilde{\Delta}(\alpha(\xi))_k) \neq 0$$

for some field k which is a $\mathcal{Z}$–algebra. By the Main Theorem, $\xi >^{\mathrm{op}}_{LR} \zeta$. □

We remark that (2.10) also holds with the order $\leq^{\mathrm{op}}_{LR}$ replaced by the dominance order $\trianglelefteq$. This is implicit in [CPS].

(2.11) **Corollary.** *The assertion of Conjecture* [DPS; (2.5.2)] *regarding a strict stratification is true for $\widetilde{S}_q(r,r)$.*

Proof. The conjecture [DPS; (2.5.2)] asserts stratification properties of an algebra $\widetilde{A}^+ = \operatorname{End}_{\widetilde{H}}(\widetilde{T}^+)$, where $\widetilde{T}^+ = \widetilde{T} \bigoplus \widetilde{X}$, $\widetilde{X}$ having certain filtration properties. In the present case, we take $\widetilde{X} = 0$, so that the strict stratification assertion of the conjecture is essentially (2.10(c)). Note that the poset $(\Xi, \leq^{\mathrm{op}}_{LR})$ is the minimal quasi-poset $(\Omega_{\min}, \leq^{\mathrm{op}}_{LR})$ associated to the quasi-poset $(\Omega, \leq^{\mathrm{op}}_{LR})$ given in [DPS; (2.5.2)]; see the remark after [DPS; (1.2.4)]. Note also that the algebra $\widetilde{S}_q(r,r)$ is Morita equivalent to the one described in [DPS; (2.5.2)]. □

If $\mathcal{Z} \to \mathcal{Z}'$ is a homomorphism of $\mathcal{Z}$ into a commutative, Noetherian ring $\mathcal{Z}'$ (e.g., a field k), let

$$\widetilde{S}_q(n,r,\mathcal{Z}') = \operatorname{End}_{\widetilde{H}_{\mathcal{Z}'}}\Big(\bigoplus_{\lambda \in \Lambda^+(n,r)} \operatorname{ind}^{\widetilde{H}_{\mathcal{Z}'}}_{\widetilde{H}_{\lambda \mathcal{Z}'}} \mathrm{IND}_\lambda\Big), \tag{2.12}$$

where $\widetilde{H}_{\mathcal{Z}'} = \widetilde{H} \otimes_{\mathcal{Z}} \mathcal{Z}'$, etc. By [DPS; (2.3.8)], $\widetilde{S}_q(n,r,\mathcal{Z}') \cong \widetilde{S}_q(n,r)_{\mathcal{Z}'}$. This fact is essentially well-known, though the proof in [DPS; (2.3.8)] is in the same spirit as that for (2.7). In particular, the above results apply to the classical q-Schur algebras over fields which are Morita equivalent to the algebras $\widetilde{S}_q(n,r,k)$.

The following remarks give two alternative proofs of the Main Theorem. Both proofs require deep properties of cells; thus, they are not as elementary as the argument given above. In preparation for this, we also explicitly identify the subposet $\Xi(n,r)$ used in the Main Theorem.

(2.13) *Remarks.* (a) It is well-known that Kazhdan–Lusztig cells in $W = \mathfrak{S}_r$ are determined by the Robinson-Schensted correspondence (see [BV]) which takes $w \in W$ to a pair $(P(w), Q(w))$ of standard tableaux of the same shape with the property that $y \sim_{LR} w$ if and only if the shape of $P(y)$ is the same as the shape of $P(w)$. (A standard tableau is a tableau of the form $\mathbf{t}^\lambda x$ ($x \in W$) which has increasing rows and columns; λ is called its shape. Here $\mathbf{t}^\lambda$ is the standard tableau of shape λ defined at the start of this section.) For $\lambda \in \Lambda^+(r)$, let λ' be the dual partition of λ, namely, the partition whose Young diagram is the transpose of $\mathcal{Y}(\lambda)$, and let $w_{0,\lambda}$ be the longest word in W_λ. Then $P(w_{0,\lambda})$ is the transpose of the standard tableau $\mathbf{t}^\lambda$; thus, we obtain a bijection $\beta_1 : \Xi \to \Lambda^+(r)$ such that, if $\beta_1(\xi) = \lambda'$, then ξ contains the longest element $w_{0,\lambda}$ of W_λ. Let $\beta : \Xi \to \Lambda^+(r)$ be defined by $\beta(\xi) = \beta_1(\xi)'$. *We claim we have an equality $\alpha = \beta$ of functions.* Indeed, as seen from remarks immediately above [DPS; (2.3.7)], we have, for any $\xi \in \Xi$, $\xi \leq_{LR} \beta^{-1}(\alpha(\xi))$. So, if ξ is maximal (relative to $\leq_{LR}$), then $\beta(\xi) = \alpha(\xi)$. Now our claim follows by induction.

Thus, we can explicitly describe the set $\Xi(n,r)$ as the set of all two-sided cells ξ containing some $w_{0,\lambda}$, $\lambda \in \Lambda^+(n,r)$.

(b) By (a), we see now that the Main Theorem follows from (2.8) if β defines a poset isomorphism $(\Xi, \leq_{LR}^{\mathrm{op}}) \xrightarrow{\sim} (\Lambda^+(r), \trianglelefteq)$. A proof of this isomorphism has been given by Shi [S]; it turns out to be a special case of a conjecture of Lusztig [L1; Conjecture D]. For the reader's convenience, we include another, shorter proof, obtained by a further consideration of the Robinson–Schensted correspondence. It suffices to prove

(2.13.1) $$w_{0,\mu} \leq_{LR} w_{0,\lambda} \Leftrightarrow \mu \trianglerighteq \lambda.$$

For $w \in W$, let $\mathcal{L}(w) = \{s \in S \mid sw < w\}$ be the left-set of w. By [Du; (3.3)] we see that $\mu \trianglerighteq \lambda$ if and only if there exists a $w \in W$ such that $\mathcal{L}(w) \supseteq \mathcal{L}(w_{0,\lambda})$ and $w_{0,\mu} \sim_L w$. This implies $w_{0,\mu} \leq_{LR} w_{0,\lambda}$ since $w \leq_R w_{0,\lambda}$. The converse implication in (2.13.1) follows from [LX; (3.2)]. (This latter result requires the positivity of the Kazhdan–Lusztig polynomials, proved using perverse sheaves in [KL2] via étale intersection cohomology. The argument in [S] also uses positivity.)

(c) By [L1; (4.1)] or [L2; (6.3)], we have for $y, w \in W$,

$$y \leq_L w \text{ and } y \sim_{LR} w \Rightarrow y \sim_L w. \tag{2.13.2}$$

As remarked by Lusztig [L1], this simple sounding result appears to require the Kazhdan–Lusztig conjecture [KL1; (8.1)], proved in [BB], [BK], and again involving perverse sheaves [KL2]. Using (2.13.2), we thus obtain a triangular decomposition

$$\widetilde{T}_{\beta(\xi)K} = \widetilde{S}_{\xi K} \oplus \bigoplus_{\zeta \in \Xi(n,r), \zeta >^{\mathrm{op}}_{LR} \xi} \widetilde{S}_{\zeta K}^{\oplus d_\zeta}.$$

Now the proof of (2.8) works with respect to the poset $(\Xi, \leq^{\mathrm{op}}_{LR})$, which also proves the Main Theorem.

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