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Borel Subalgebras Redux with Examples from Algebraic and Quantum Groups

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Abstract. Both building upon and revising previous literature, this paper formulates the general notion of a Borel subalgebra B of a quasi-hereditary algebra A. We present various general constructions of Borel subalgebras, establish a triangular factorization of A, and relate the concept to graded Kazhdan–Lusztig theories in the sense of Cline *et al.* (*Tôhoku Math. J.* **45** (1993), 511–534). Various interesting types of Borel subalgebras arise naturally in different contexts. For example, 'excellent' Borel subalgebras come about by abstracting the theory of Schubert varieties. Numerous examples from algebraic groups, q-Schur algebras, and quantum groups are considered in detail.

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This paper casts the theory of Borel subalgebras in the context of quasi-hereditary algebras. The latter arise in several natural contexts. For example, their module categories often appear in connection with the representation theory of algebraic groups over fields of positive characteristic or of quantum groups at a root of unity. Classical Schur algebras attached to the general linear groups GL_n as well as their quantized versions associated to the quantum general linear groups provide typical examples of quasi-hereditary algebras arising in this way. There are many other algebraic examples in Lie theory. A further important source of quasi-hereditary algebras comes from geometry, in the study of perverse sheaves. For example, the perverse sheaf category on a suitably nice stratified topological space X is the module category for a quasi-hereditary algebra. (See [PS1, PS2].)

Green [G] gave the first example of a Borel subalgebra of a quasi-hereditary algebra, though he did not offer a general definition of the concept. Later, König [K1] introduced the notion of an *exact* Borel subalgebra, which did not fit Green's example, but did apply for the category $\mathcal{O}$ associated to a semisimple complex Lie

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algebra. The second author of this paper introduced in [K1, Appendix] a further notion, which synthesized these two cases,⋆ but which still does not obviously capture other interesting examples, such as those given in [Dy] and [DR].

The notion of a *Borel subalgebra*, introduced in Section 2, does seem to capture all the known examples, and it is sufficiently rich to have an interesting theory. The original notion of [K1, Appendix] is recast, adapting related terminology of Woodcock [W1, W2], in Section 4 as a *homological Borel subalgebra*, and several new and stronger variations are proposed. All these variations mirror to some extent properties of Borel subgroups of algebraic groups, discovered by van der Kallen and Polo [vdK]. We also demonstrate (in Section 5) that König's notion of an exact Borel subalgebra is interesting in the context of infinitesimal groups and quantum groups at a root of unity.

A key ingredient in our point of view centers on a categorical approach. Thus, we first define various *Borel categories* associated to a highest weight category (i.e., the module category of a quasi-hereditary algebra). Then we consider natural constructions, such as passing to endomorphism algebras, which give rise to subalgebras of quasi-hereditary algebras. In some cases, this process is interesting and nontrivial. (See, for instance, (5.2) where the issue centers on the injectivity of a certain algebra homomorphism.) In addition to naturality, the systematic use of the categorical approach has the advantage of making some of the discussion invariant under Morita equivalence, quite unusual for subalgebra results.

We organize our paper as follows. Section 1 reviews some preliminary material. Section 2 takes up the idea of a *Borel category* $\mathcal{B}$ (and a Borel subalgebra B) associated to a highest weight category $\mathcal{C}$ (and its associated quasi-hereditary algebra A). The axioms imply that tensor induction $A \otimes_B -$ of some irreducible B-modules give standard modules for A, while the other irreducible modules induce to the zero module. Remarkably, in our formulation every quasi-hereditary algebra has at least one Borel subalgebra, cf. (2.13). Several general constructions and a triangular factorization are discussed. Of course, some Borel subalgebras may be more interesting than others.

Section 3 continues this discussion by considering the existence of Borel categories specific to highest weight categories having a graded Kazhdan–Lusztig theory (in the sense of [CPS4]). The main result here, given in (3.5), improves upon an unpublished result due to the first two authors and E. Cline (which was itself inspired by an earlier variation by Dyer [Dy]): Quasi-hereditary algebras A, even basic algebras, with a *graded* Kazhdan–Lusztig theory [CPS4, §3, Appendix] always have natural subalgebras B that are always tightly graded Borel subalgebras.

⋆ Unfortunately, the two cases do not appear as closely related as [K1, Appendix] suggests. The second author wishes to acknowledge here that his proof of [K1, Appendix, Thm. F] on the existence of exact Borel subalgebras is incorrect, and no repair seems likely. The construction does, however, lead in that context to Borel subalgebras having only 'dominant' weights; see Section 2. One can also choose a homological (or even excellent) Borel subalgebra in an appropriately constructed generalized Schur (or q-Schur algebra – see Section 6), but one is *not* guaranteed an exact Borel subalgebra, as claimed in [K1, Appendix, Thm. E]. See remarks below (2.15).

We point out that, conversely, the existence of a suitably strong Borel subalgebra (for certain quasi-hereditary algebras) is equivalent to the truth of the Lusztig conjecture for the characters of certain irreducible rational modules for an algebraic group in positive characteristic. (See (3.9).) This discussion provides a substitute for a theory proposed by König [K3] (see the announcement [K2, p. 604]) in the exact Borel subalgebra case.

In Section 4, we define homological, exact, and excellent Borel categories. 'Homological' Borel categories capture 'Kempf's vanishing theorem' for reductive groups, while the 'excellent' theory abstracts from even stronger properties in Schubert theory. Section 4 contains a number of further results. For example, (4.8), taken together with the results of [vdK] discussed in (5.6), provides, a new characterization of the category of rational G-modules as a full subcategory of the category of rational B-modules. (Here G is a reductive group and B is a Borel subgroup of G). Using Section 6, a similar result can be obtained for quantum enveloping algebras.

Given a highest weight category $\mathcal{C} = A$-mod, the existence of a *subalgebra* B of A realizing a Borel category of a given type as B-mod is, in general, difficult to establish. Section 5 takes up this question. For example, a pleasant consequence, given in (5.2), of the axioms for an excellent Borel category is that they guarantee fairly quickly the existence of an associated subalgebra. This section concludes with a discussion of several classical situations. In (5.4), we show that exact Borel subalgebras exist for the various 'interval' quasi-hereditary algebras attached to the group schemes G_rT, the pull-back through the rth power of the Frobenius morphism of a maximal split torus T of a reductive group G. Although our methods are different here than those of [K1], our result mirrors a similar result of his for the category $\mathcal{O}$ associated to a complex semisimple Lie algebra. Exact Borel subalgebras in the infinitesimal case provide a satisfactory substitute in König's theory [K2] for those thought there to be present in generalized Schur algebras. Example (5.5) is the quantization of (5.4), while (5.6) takes up the existence of excellent Borel subalgebras for those quasi-hereditary algebras associated to a reductive algebraic group.

Finally, Section 6 considers the existence of Borel subalgebras for classical and generalized q-Schur algebras. Thus, (6.2) and (6.4) lay out the necessary q-version of [vdK] – these results follow rather quickly using [W2]. After defining generalized q-Schur algebras,* (6.10) establishes that these algebras have excellent Borel subalgebras. We conclude by considering the situation of classical q-Schur algebras $S_q(n, r)$. In this case, all the algebras involved have concrete realizations in terms of (Manin's) quantum matrix space.

* Generalized q-Schur algebras were first considered in [DS1]. To some extent they are 'concrete' realizations of quasi-hereditary algebras coming out of standard highest weight theory (but usually only given up to Morita equivalence). We give a somewhat different (but essentially equivalent) development in Section 6.

This paper suggests several further interesting directions. First, we raise here the question of the existence of excellent Borel subalgebras for some of the quasi-hereditary algebras not directly associated to algebraic or quantum groups, especially those which play an important role in the nondescribing characteristic representation theory of finite groups of Lie type in type different from type A (e.g., those announced in [DS2]). One speculates that such a theory would require some kind of 'Schubert theory' for these algebras (along the lines of [vdK], e.g., do there exist suitable 'Joseph modules'?). Even in type A, a *combinatorial* proof that the usual Borel subalgebras of q-Schur algebras are excellent (or even homological) does not presently exist. The close connection between the representation theory of q-Schur algebras $S_q(n, r)$ and the nondescribing representation theory of finite general groups $\mathrm{GL}_n(q)$ suggests another problem. It would be interesting to reintepret the existence of excellent Borel subalgebras of $S_q(n, r)$ – quite a deep fact – in terms of the nondescribing representation theory of $\mathrm{GL}_n(q)$, perhaps making use of the Morita equivalence presented in [CPS9, (9.17)]. Finally, for those quasi-hereditary algebras connected to perverse sheaves, it would be interesting to interpret geometrically the existence of various types of Borel subalgebras.

1. Preliminaries

Let be a fixed field, which for convenience is algebraically closed. We will work with various Abelian categories $\mathcal{C}$ over. Assume that there is given a set $\Lambda \equiv \Lambda(\mathcal{C})$ – the set of *weights* of $\mathcal{C}$ – indexing the distinct isomorphism classes of simple objects of $\mathcal{C}$. For $\lambda \in \Lambda$, write $L(\lambda)$ for a fixed representative from the corresponding isomorphism class of irreducible objects. The category $\mathcal{C}$ is usually assumed to have both enough injective and enough projective objects. Let $I(\lambda)$ (resp., $P(\lambda)$) denote the injective hull (resp., projective cover) of $L(\lambda)$. When several categories are in play at the same time, write $L(\mathcal{C}, \lambda)$, $P(\mathcal{C}, \lambda)$, etc. in place of $L(\lambda)$, $P(\lambda)$, etc.

If $\mathcal{C}$ is a K-finite category having finitely many simple objects, then $\mathcal{C} \cong A$-mod for a (nonuniquely) determined finite-dimensional algebra A over. In the case $\mathcal{C} = A$-mod (finite-dimensional, left modules) or mod-A (finite-dimensional, right modules) for a K-algebra A, we may write $L(A, \lambda)$, $P(A, \lambda)$, etc. in place of $L(\lambda)$, $P(\lambda)$, etc.

We will consider *highest weight categories* $\mathcal{C}$ over, in the sense of [CPS1]. In this case, Λ comes equipped with a poset structure $\leqslant$. *We will assume that $\mathcal{C}$ is K-finite with finite weight poset Λ unless explicitly stated otherwise.*⋆ Let $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) denote the standard (resp., costandard) object in $\mathcal{C}$ associated to λ. Thus, $L(\lambda)$ is isomorphic to the head (resp., socle) of $\Delta(\lambda)$ (resp., $\nabla(\lambda)$). Fixed

⋆ The finiteness of Λ is assumed only for convenience. Some K-finite highest weight categories with infinite weight posets do appear naturally in the representation theory of algebraic groups and quantum groups. See (5.4)–(5.6) and Section 6. We are able, effectively, to use the same, or slightly modified, definitions for these categories.

$\lambda \in \Lambda$, $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) can be characterized as the largest quotient (resp., subobject) of the projective cover $P(\lambda)$ (resp., injective envelope $I(\lambda)$) of $L(\lambda)$ having composition factors isomorphic to $L(\mu)$ with $\mu \leqslant \lambda$. $P(\lambda)$ (resp., $I(\lambda)$) has a decreasing (resp., increasing) filtration with top (resp., bottom) section isomorphic to $\Delta(\lambda)$ (resp., $\nabla(\lambda)$) and with lower (resp., higher) sections isomorphic to $\Delta(\mu)$ (resp., $\nabla(\mu)$) for some $\mu > \lambda$.

A module category $\mathcal{C} = A$-mod for a finite-dimensional algebra A is a highest weight category (relative to some poset structure on Λ) if and only if A is quasi-hereditary [CPS1, (3.6)]. Given the quasi-hereditary algebra A, the standard and costandard objects in $\mathcal{C} = A$-mod depend on the chosen poset structure $\leqslant$ on Λ. To emphasize this fact, write $\Delta(A, \leqslant, \lambda)$, $\nabla(A, \leqslant, \lambda)$ for these objects as a way of indicating their dependence on $\leqslant$.

An ideal Γ (resp., coideal Ω) in the weight poset Λ of a highest weight category $\mathcal{C}$ determines a highest weight category $\mathcal{C}[\Gamma]$ (resp., $\mathcal{C}(\Omega)$) with weight poset Γ (resp., Ω), cf. [CPS1, §3] for definitions. Let i^* and $i^!$ be the left and right adjoints of the inclusion functor i_*: $\mathcal{C}[\Gamma] \to \mathcal{C}$. Similarly, the quotient functor j^*: $\mathcal{C} \to \mathcal{C}(\Omega)$ has left adjoint $j_!$ and right adjoint j_* which both define sections for j^* (i.e., $j^* j_! \cong \mathrm{id}_{\mathcal{C}(\Omega)} \cong j^* j_*$).

The proof of the following lemma is immediate.

(1.1) LEMMA. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, $\mathcal{S}$ and $\mathcal{T}$ be Serre subcategories of $\mathcal{C}$ and $\mathcal{D}$, respectively. Let Ψ: $\mathcal{C} \to \mathcal{D}$ and Φ: $\mathcal{D} \to \mathcal{C}$ be additive functors such that Ψ carries $\mathcal{S}$ to $\mathcal{T}$ and Φ carries $\mathcal{T}$ to $\mathcal{S}$. Assume that the canonical quotient functor J^*: $\mathcal{C} \to \mathcal{C}/\mathcal{S}$ (resp., j^*: $\mathcal{D} \to \mathcal{D}/\mathcal{T}$) admits a right (resp., left) adjoint J_* (resp., $j_!$) satisfying $J^* J_* \cong \mathrm{id}_{\mathcal{C}/\mathcal{S}}$ (resp., $j^* j_! \cong \mathrm{id}_{\mathcal{D}/\mathcal{T}}$). Let $\overline{\Psi}$: $\mathcal{C}/\mathcal{S} \to \mathcal{D}/\mathcal{T}$ and $\overline{\Phi}$: $\mathcal{D}/\mathcal{T} \to \mathcal{C}/\mathcal{S}$ be the induced functors on quotient categories. Suppose that Φ is left adjoint to Ψ. Then $\overline{\Phi}$ is left adjoint to $\overline{\Psi}$.*

For a fixed highest weight category $\mathcal{C}$ with poset $(\Lambda, \leqslant)$, consider the various poset structures $\leqslant'$ on Λ relative to which $\mathcal{C}$ is a highest weight category having the same ∇-objects and Δ-objects as defined by $\leqslant$. Among these $\leqslant'$, there is a unique minimal partial ordering $\leqslant_{\mathrm{min}}$, generated by the preorder $\leqslant_{\mathrm{pre}}$ on Λ obtained by putting $\lambda \leqslant_{\mathrm{pre}} \mu$ if and only if $L(\lambda)$ is a composition factor of $\nabla(\mu)$ or of $\Delta(\mu)$. The multiplicity equalities $[P(\lambda) : \Delta(\mu)] = [\nabla(\mu) : L(\lambda)]$ and $[I(\lambda) : \nabla(\mu)] = [\Delta(\mu) : L(\lambda)]$ given in [CPS1, (3.11)] show that each $P(\lambda)$ (resp., $I(\lambda)$) has a filtration in which the $\Delta(\mu)$-sections (resp., $\nabla(\mu)$-sections) are correctly ordered relative to $\leqslant_{\mathrm{min}}$.

It is often possible to generate $\leqslant_{\mathrm{min}}$ from the preorder $\leqslant'_{\mathrm{pre}}$ defined by putting $\mu \leqslant'_{\mathrm{pre}} \lambda$ provided the multiplicity $[\nabla(\lambda) : L(\mu)]$ of $L(\mu)$ as a composition factor of $\nabla(\lambda)$ is not zero. Obviously, this occurs when the following condition holds:

(1.2) HYPOTHESIS. For any $\lambda, \mu \in \Lambda$, if $[\Delta(\lambda) : L(\mu)] \neq 0$, then $[\nabla(\lambda) : L(\mu)] \neq 0$.

Reversing the roles of the Δ and ∇-objects in (1.2). However, (1.2) will be sufficient for our purposes. Both (1.2) and $(1.2)^{\mathrm{op}}$ hold provided $\mathcal{C}$ has a strong duality [CPS2], so that any $\Delta(\lambda)$ has the same image in the Grothendieck group of $\mathcal{C}$ as $\nabla(\lambda)$. Another important situation in which (1.2) holds occurs when A is realized as $A = \widetilde{A}\otimes_{\mathcal{O}}$, where $\widetilde{A}$ is an integral $\mathcal{O}$-quasi-hereditary algebra [CPS3] for a (local) commutative ring $\mathcal{O}$ with A_K split semisimple over the quotient field K of $\mathcal{O}$. For the argument, see [DPS1, 2].*

A K-finite category $\mathcal{C}$ is (left) *directed* with respect to a poset structure $\leqslant$ on its weight set Λ provided $\mathrm{Ext}^1_{\mathcal{C}}(L(\lambda), L(\mu)) \neq 0 \Rightarrow \lambda < \mu$. If $\mathcal{C}$ is directed by $\leqslant$, it is a highest weight category with poset $(\Lambda, \leqslant)$, standard objects $\Delta(\lambda) \cong L(\lambda)$, and costandard objects $\nabla(\lambda) \cong I(\lambda)$. Any Serre subcategory $\mathcal{S}$ of a directed category $\mathcal{C}$ is directed by restricting $\leqslant$ to the weight set $\Lambda' \subseteq \Lambda$ of $\mathcal{S}$. The quotient category $\mathcal{C}/\mathcal{S}$ is directed by restricting $\leqslant$ to the weight set $\Lambda\backslash\Lambda'$ of $\mathcal{C}/\mathcal{S}$.

The ∇- and Δ-objects in a general highest weight category $\mathcal{C}$ have simple homological characterizations: For $M \in \mathrm{Ob}(\mathcal{C})$ and $\lambda \in \Lambda$,

$$\dim \mathrm{Ext}^n_{\mathcal{C}}(M, \nabla(\nu)) = \delta_{n,0}\delta_{\lambda,\nu}, \quad \forall n \geqslant 0,\ \nu \in \Lambda \Leftrightarrow M \cong \Delta(\lambda). \tag{1.3}$$

See, e.g., [CPS8, (3.2.1)]. A similar dual statement characterizes ∇-objects.

Let $D^b(\mathcal{C})$ be the bounded derived category of $\mathcal{C}$. A variation on (1.3) remains valid for some objects in $D^b(\mathcal{C})$. Let $D^{b,\leqslant 0}(\mathcal{C})$ (resp., $D^{b,\geqslant 0}(\mathcal{C})$) denote the full subcategory of $D^b(\mathcal{C})$ having objects isomorphic to complexes concentrated in nonpositive (resp., nonnegative) degrees.

(1.4) LEMMA. (1) *If $X \in \mathrm{Ob}(D^{b,\leqslant 0}(\mathcal{C}))$ and $\mathrm{Hom}^n_{D^b(\mathcal{C})}(X, \nabla(\nu)) = 0$, for all $\nu \in \Lambda$ and all $n > 0$, then X is isomorphic to an object in $\mathcal{C}$ which has a filtration with sections of the form $\Delta(\lambda)$, $\lambda \in \Lambda$.*

(2) *If $X \in \mathrm{Ob}(D^{b,\geqslant 0}(\mathcal{C}))$ and $\mathrm{Hom}^n_{D^b(\mathcal{C})}(\Delta(\nu), X) = 0$ for all $\nu \in \Lambda$ and all $n > 0$, then X is isomorphic to an object in $\mathcal{C}$ which has a filtration with sections of the form $\nabla(\lambda)$, $\lambda \in \Lambda$.*

Proof. See a similar argument in [CPS5, (1.2)]. □

(1.5) LEMMA. *Let $\lambda \in \Lambda$. Then:*

(1) *Any $M \in \mathrm{Ob}(\mathcal{C})$ is a nonzero homomorphic image of $\Delta(\lambda)$ if and only if $\dim \mathrm{Hom}_{\mathcal{C}}(M, \nabla(\mu)) = \delta_{\lambda,\mu}$ for all $\mu \in \Lambda$.*
(2) *Any $M \in \mathrm{Ob}(\mathcal{C})$ is a submodule of $\nabla(\lambda)$ if and only if $\dim \mathrm{Hom}_{\mathcal{C}}(\Delta(\mu), M) = \delta_{\lambda,\mu}$ for all $\mu \in \Lambda$.*

Proof. We will prove (1); the dual argument for (2) is similar. Any nonzero epimorphic image M of $\Delta(\lambda)$ satisfies the stated dimension condition. Conversely,

* The hypothesis that $\mathcal{O}$ have Krull dimension $\leqslant 2$ is assumed in [DPS1] in order to guarantee the existence of certain lattices. Here, we can use the fact that the $\widehat{\mathcal{O}}$-versions of $\Delta(\lambda)$, $\nabla(\lambda)$ are both $\widehat{\mathcal{O}}$-free lattices in the same irreducible $A_{\widehat{K}}$-module, where $\widehat{\mathcal{O}}$ is the completion of $\mathcal{O}$, $\widehat{K}$ is the quotient field of $\widehat{\mathcal{O}}$. Passage to $\widehat{\mathcal{O}}$ is required to assume that $\widehat{\mathcal{O}}$-versions of $\Delta(\lambda)$ and $\nabla(\lambda)$ exist.

suppose the dimension condition holds. If M has a composition factor $L(\nu)$ with $\nu \not\leqslant \lambda$, then there is a nonzero morphism $M \to I(\nu)$. But $I(\nu)$ has a filtration with bottom section $\nabla(\nu)$ and higher sections $\nabla(\omega)$, $\omega > \nu$. So $\mathrm{Hom}_{\mathcal{C}}(M, I(\nu)) = 0$, a contradiction. Thus, all the composition factors $L(\nu)$ of M satisfy $\nu \leqslant \lambda$. Since M has head $L(\lambda)$, M is an epimorphic image of $\Delta(\lambda)$. □

2. Borel Categories and Subalgebras

Let $\mathcal{C}$ be a fixed highest weight category with finite weight poset $(\Lambda^+, \leqslant^+)$. The "+"-notation is chosen because we view Λ^+ as analogous to a set of "dominant weights". We consider various "Borel categories" for $\mathcal{C}$ (as well as associated algebras), taking up first the most general notion of a Borel category associated to $\mathcal{C}$. These Borel categories and their dual variation occur commonly in algebraic and quantum groups (as well as in some Morita equivalent contexts). Although we are ultimately interested in algebras and their module categories, we obtain the best perspective and flexibility by setting up a categorical framework.

(2.1) LEMMA. *Let $\Psi\colon \mathcal{C} \to \mathcal{B}$ be an exact, additive functor from $\mathcal{C}$ to a highest weight category $\mathcal{B}$ with weight poset $(\Lambda, \leqslant)$. Assume that Λ^+ is a subset of Λ, and that $\leqslant^+$ is compatible with $\leqslant$ (i.e., $\lambda \leqslant^+ \mu \Rightarrow \lambda \leqslant \mu$ for all $\lambda, \mu \in \Lambda^+$). Assume that Ψ admits a left adjoint $\Psi_!\colon \mathcal{B} \to \mathcal{C}$. The following three statements are equivalent:*

(1) *For $\lambda \in \Lambda^+$, $\Psi\nabla(\mathcal{C}, \lambda)$ is isomorphic to a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$.*
(2) *For $\mu \in \Lambda^+$, $\Psi_!\Delta(\mathcal{B}, \mu)$ is a nonzero homomorphic image of $\Delta(\mathcal{C}, \mu)$, and $\Psi_!\Delta(\mathcal{B}, \mu) = 0$ if $\mu \in \Lambda\backslash\Lambda^+$.*
(3) *For $\mu \in \Lambda$, we have*

$$\Psi_!\Delta(\mathcal{B}, \mu) \cong \begin{cases} \Delta(\mathcal{C}, \mu), & \mu \in \Lambda^+, \\ 0, & \mu \notin \Lambda^+. \end{cases}$$

Proof. (1) $\Leftrightarrow$ (2): By adjointness, for $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we have

$$\mathrm{Hom}_{\mathcal{C}}(\Psi_!\Delta(\mathcal{B}, \mu), \nabla(\mathcal{C}, \lambda)) \cong \mathrm{Hom}_{\mathcal{B}}(\Delta(\mathcal{B}, \mu), \Psi\nabla(\mathcal{C}, \lambda)).$$

By (1.5), (1) implies that $\dim \mathrm{Hom}_{\mathcal{C}}(\Psi_!\Delta(\mathcal{B}, \mu), \nabla(\mathcal{C}, \lambda)) = \delta_{\lambda\mu}$, and (2) follows. Conversely, (2) implies $\dim \mathrm{Hom}_{\mathcal{B}}(\Delta(\mathcal{B}, \mu), \Psi\nabla(\mathcal{C}, \lambda)) = \delta_{\lambda\mu}$, and (1) follows.

(3) $\Rightarrow$ (2) is trivial.

(1) $\Rightarrow$ (3): Since now (2) holds, it suffices to show that there is a surjective homomorphism $\Psi_!\Delta(\mathcal{B}, \lambda) \to \Delta(\mathcal{C}, \lambda)$, for all $\lambda \in \Lambda^+$. We first claim:

($*$) If $\lambda \in \Lambda^+$, then $\Psi L(\mathcal{C}, \lambda)$ has socle isomorphic to $L(\mathcal{B}, \lambda)$, while all other composition factors $L(\mathcal{B}, \nu)$ satisfy $\nu < \lambda$.

If $\lambda \in \Lambda^+$ is minimal (with respect to $\leqslant^+$), then $\nabla(\mathcal{C}, \lambda) \cong L(\mathcal{C}, \lambda)$, so that $\Psi L(\mathcal{C}, \lambda) \cong \Psi\nabla(\mathcal{C}, \lambda)$ is a nonzero subobject of $\nabla(\mathcal{B}, \lambda)$, and ($*$) holds in this case. If λ is not minimal, induction implies that $(*)$ holds for any composition factor of $\nabla(\mathcal{C},\lambda)/L(\mathcal{C},\lambda)$. Since $0 \neq \Psi\nabla(\mathcal{C},\lambda) \subseteq \nabla(\mathcal{B},\lambda)$, $\Psi L(\mathcal{C},\lambda)$ has socle $L(\mathcal{B},\lambda)$. The other composition factors $L(\mathcal{B},\nu)$ must satisfy $\mu < \lambda$. So $(*)$ holds for all $\lambda \in \Lambda^+$.

The exactness of Ψ and $(*)$ imply, for all $\lambda \in \Lambda^+$, that $L(\mathcal{B},\lambda)$ occurs with multiplicity one as a composition factor of $\Psi\Delta(\mathcal{C},\lambda)$, while all other composition factors $L(\mathcal{B},\nu)$ satisfy $\nu < \lambda$. By highest weight theory for $\mathcal{B}$, there is a nonzero map $\Delta(\mathcal{B},\lambda) \to \Psi\Delta(\mathcal{C},\lambda)$ which remains nonzero upon composition with the natural surjection $\Psi\Delta(\mathcal{C},\lambda) \to \Psi L(\mathcal{C},\lambda)$. By adjointness, there is a nonzero map $\Psi_!\Delta(\mathcal{B},\lambda) \to \Delta(\mathcal{C},\lambda)$ remaining nonzero upon composition with $\Delta(\mathcal{C},\lambda) \twoheadrightarrow L(\mathcal{C},\lambda)$. Hence, $\Psi_!\Delta(\mathcal{B},\lambda)$ maps surjectively onto $\Delta(\mathcal{C},\lambda)$, as required. □

(2.2) DEFINITION. A *Borel category* associated to the highest weight category $\mathcal{C}$ is a highest weight category $\mathcal{B}$ with weight poset $(\Lambda,\leqslant)$, together with a exact, additive functor $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ satisfying the following conditions:

- (B0) The functor Ψ admits a left adjoint $\Psi_!\colon\ \mathcal{B} \to \mathcal{C}$.
- (B1) $\Lambda^+ \subseteq \Lambda$ and $\leqslant^+$ is compatible with $\leqslant$.
- (B2) For $\lambda \in \Lambda^+$, $\Psi\nabla(\mathcal{C},\lambda)$ is isomorphic to a nonzero subobject of $\nabla(\mathcal{B},\lambda)$.
- (B3) For $\lambda \in \Lambda^+$, $\Delta(\mathcal{B},\lambda) \cong L(\mathcal{B},\lambda)$. There is a second poset structure $\leqslant'$ on Λ making $\mathcal{B}$ into a directed category.

We often say that $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ defines a Borel category, or that $\mathcal{B}$ is a Borel category of $\mathcal{C}$.

(2.3) *Remarks*. (1) (2.1) implies that (B2) can be replaced by (2.1(3)). If Ψ: $\mathcal{C} \to \mathcal{B}$ defines a Borel category of $\mathcal{C}$, an induction based on $(*)$ in the proof of (2.1) leads to the following properties of Ψ: (a) $\Psi V \neq 0$ for any nonzero object V in $\mathcal{C}$; and (b) $\Psi(\varphi) \neq 0$ for any nonzero morphism φ in $\mathcal{C}$. If $\mathcal{C}$ and $\mathcal{B}$ are realized as A-mod and B-mod for quasi-hereditary algebras A and B, respectively, and Ψ is the pull-back functor defined by an algebra homomorphism $\iota\colon\ B \to A$ (see (2.6) below), these properties are trivial.

(2) (B3) implies that $\Psi_! L(\mathcal{B},\lambda) \cong \Delta(\mathcal{C},\lambda)$ if $\lambda \in \Lambda^+$ and is 0 otherwise. The result on $\lambda \notin \Lambda^+$ follows since $\Psi_!\Delta(\mathcal{B},\lambda) = 0$ and $\Psi_!$ is right exact.

(3) We can often assume $\leqslant' = \leqslant$. Since the highest weight category structure on $\mathcal{B}$ defined by the poset structure $\leqslant'$ on Λ is directed, $\nabla(\mathcal{B},\leqslant,\lambda) \subseteq \nabla(\mathcal{B},\leqslant',\lambda)$. Thus, if (1.2) holds for $\mathcal{C}$, then (B2) implies that $\leqslant^+_{\min}$ is compatible with $\leqslant'$. Replacing $\leqslant^+$ with $\leqslant^+_{\min}$ (keeping the same standard and costandard modules), the highest weight category structure on $\mathcal{B}$ defined by $\leqslant'$ also defines $\mathcal{B}$ as a Borel category for $\mathcal{C}$. We can replace $\leqslant$ with $\leqslant'$ to obtain a Borel category $\mathcal{B}$ with poset $(\Lambda,\leqslant)$ such that $\leqslant$ directs $\mathcal{B}$. The replacement of $\leqslant$ with $\leqslant'$, unlike the replacement of $\leqslant^+$ with $\leqslant^+_{\min}$, may change the structure of $\mathcal{B}$ as a highest weight category, i.e., $\Delta(\mathcal{B},\leqslant',\lambda)$ and $\nabla(\mathcal{B},\leqslant',\lambda)$ may differ from $\Delta(\mathcal{B},\leqslant,\lambda)$ and $\nabla(\mathcal{B},\leqslant,\lambda)$. Additional structure on $\mathcal{B}$ with poset $(\Lambda,\leqslant)$ (see Section 4) may

not exist on $\mathcal{B}$ with poset $(\Lambda, \leqslant')$, and *vice versa*. This is one reason to have different partial orderings on Λ in the definition of Borel categories. In practice, even when (1.2) does hold, two distinct poset structures on Λ can arise naturally, so it is useful to allow some flexibility in (2.2).

(4) Assume the set-up in (2.2), but that $\leqslant$ directs $\mathcal{B}$. Then $\mathcal{B}$ can be replaced by any full Abelian subcategory $\mathcal{B}'$ containing the image of $\Psi\colon \mathcal{C} \to \mathcal{B}$, all objects $L(\mathcal{B}, \lambda)$ for $\lambda \in \Lambda^+$, and having enough injectives. In this case, the poset Λ' for $\mathcal{B}'$ consists of the subposet of Λ corresponding to the irreducible objects in $\mathcal{B}'$.

A common theme in the theory of highest weight categories $\mathcal{C}$ concerns relating a property of $\mathcal{C}$ to a similar property for the highest weight categories $\mathcal{C}[\Gamma]$ and $\mathcal{C}(\Lambda\backslash\Gamma)$ for an ideal Γ in the weight poset Λ of $\mathcal{C}$. The following result shows that the existence of Borel categories carries over to the categories $\mathcal{C}[\Gamma]$ and $\mathcal{C}(\Lambda\backslash\Gamma)$. We will return to this theme in Sections 4 and 5; see (4.9) and (5.3).

(2.4) PROPOSITION. *Let $\Psi\colon \mathcal{C} \to \mathcal{B}$ define the highest weight category $\mathcal{B}$, with weight poset $(\Lambda, \leqslant)$, as a Borel category of the highest weight category $\mathcal{C}$. Let Γ^+ be an ideal in $(\Lambda^+, \leqslant^+)$, and Γ an ideal in $(\Lambda, \leqslant)$ satisfying $\Gamma \cap \Lambda^+ = \Gamma^+$. Then:*

(1) *The restriction $\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ of Ψ defines $\mathcal{B}[\Gamma]$ as a Borel category of $\mathcal{C}[\Gamma^+]$.*

(2) *$\Psi\colon \mathcal{C} \to \mathcal{B}$ induces a functor $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$ of quotient categories, defining $\mathcal{B}(\Lambda\backslash\Gamma)$ as a Borel category of $\mathcal{C}(\Lambda^+\backslash\Gamma^+)$.*

Proof. (1) Condition (B2), together with the exactness of Ψ, shows, for $\lambda \in \Gamma^+$, that $\Psi L(A, \lambda) \subseteq \nabla(B, \lambda)$ is an object of $\mathcal{B}[\Gamma]$. So, Ψ induces a functor $\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$.

Condition (2.1(3)), which is equivalent to condition (B2), insures that $\Psi_!\Delta(B, \mu) \in \mathrm{Ob}(\mathcal{C}[\Gamma^+])$ for all $\mu \in \Gamma$, since $\Gamma^+ = \Gamma \cap \Lambda^+$. Having an exact right adjoint, $\Psi_!$ is right exact, so $L(B, \mu)$, as a quotient object of $\Delta(B, \mu)$, is sent by $\Psi_!$ into $\mathcal{C}[\Gamma^+]$. This gives a functor $\Psi_!|_{\mathcal{B}[\Gamma]}\colon \mathcal{B}[\Gamma] \to \mathcal{C}[\Gamma^+]$, serving as a left adjoint to $\Psi|_{\mathcal{C}[\Gamma^+]}$.

Conditions (B1)–(B3) are satisfied. (Note that $\mathcal{B}[\Gamma]$ is a Serre subcategory of $\mathcal{B}$, and is directed by $\leqslant'$.) So (1) is proved.

(2) Since Ψ (resp., $\Psi_!$) sends $\mathcal{C}[\Gamma^+]$ (resp., $\mathcal{B}[\Gamma]$) into $\mathcal{B}[\Gamma]$ (resp., $\mathcal{C}[\Gamma^+]$), it induces a functor $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$ (resp., $\overline{\Psi}_!\colon \mathcal{B}(\Lambda\backslash\Gamma) \to \mathcal{C}(\Lambda^+\backslash\Gamma^+)$). By (1.1), $\overline{\Psi}_!$ is left adjoint to $\overline{\Psi}$, and (B1)–(B3) are satisfied. □

The following lemma will be used in the proof of (2.15) and (4.6).

(2.5) LEMMA. *Let $\Psi\colon \mathcal{C} \to \mathcal{B}$ define the highest weight category $\mathcal{B}$, with weight poset $(\Lambda, \leqslant)$, as a Borel category of $\mathcal{C}$. Suppose that $\leqslant$ directs $\mathcal{B}$. Let Γ is a subset of $\Lambda\backslash\Lambda^+$, and $\mathcal{S}$ be the Serre subcategory of $\mathcal{B}$ generated by the $L(B, \mu)$ with $\mu \in \Gamma$. Let $j^*\colon \mathcal{B} \to \mathcal{B}/\mathcal{S}$ be the canonical quotient functor. Then $j^*\Psi\colon \mathcal{C} \to \mathcal{B}/\mathcal{S}$ defines $\mathcal{B}/\mathcal{S}$, with weight poset $(\Lambda\backslash\Gamma, \leqslant|_{\Lambda\backslash\Gamma})$, as a Borel category of $\mathcal{C}$.*

Proof. Since $\leqslant$ directs $\mathcal{B}$, it also directs $\mathcal{B}/\mathcal{S}$. Now $j^*\Psi$ has a left adjoint $\Psi_! j_!$. The conditions (B1) and (B3) are obvious. Since $\Lambda^+ \cap \Gamma = \emptyset$, $j^*\Psi\nabla(\mathcal{C}, \lambda) \neq 0$ for $\lambda \in \Lambda^+$. Also, (B2) holds, since, for $\lambda \in \Lambda^+$,

$$j^*\Psi\nabla(\mathcal{C}, \lambda) \subseteq j^*\nabla(\mathcal{B}, \lambda) \cong j^*I(\mathcal{B}, \lambda) \cong I(\mathcal{B}/\mathcal{S}, \lambda) \cong \nabla(\mathcal{B}/\mathcal{S}, \lambda). \qquad \square$$

To reformulate the notion of a Borel category in terms of algebras, suppose that $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines a Borel category. We wish to realize $\mathcal{C}$ and $\mathcal{B}$ as module categories A-mod and B-mod, respectively, so that Ψ is defined by an algebra homomorphism $\iota\colon B \to A$. (In this case, we think of the pair (B, ι) as a *prealgebra* of A – of course, in ideal situations, ι will be injective and so identify B with a subalgebra of A.)

More generally, if $\iota\colon B \to A$ is a homomorphism of finite-dimensional algebras, let $\iota^*\colon A\text{-mod} \to B\text{-mod}$ be the pull-back functor (i.e., if M is an A-module, then ι^*M is the B-module obtained by making B act on M through the morphism ι). Now ι^* admits a right adjoint $\iota_* = \mathrm{Hom}_B(A, -)$ and a left adjoint $\iota_! = A \otimes_B -$.

(2.6) PROPOSITION. *Let $\Psi\colon \mathcal{C} \to \mathcal{B}$ define $\mathcal{B}$ as a Borel category. Let P be a projective generator for $\mathcal{B}$, and set*

$$B = \mathrm{End}_{\mathcal{B}}(P)^{\mathrm{op}}, \qquad A = \mathrm{End}_{\mathcal{C}}(\Psi_! P)^{\mathrm{op}}.$$

Let $\iota\colon B \to A$ be defined by $\iota(b) = \Psi_!(b)$. Then:

(1) *$Q = \Psi_! P$ is a projective generator for $\mathcal{C}$.*
(2) *The algebras A and B are quasi-hereditary.*
(3) *Identifying $\mathcal{C}$ to A-mod by means of the equivalence $\mathrm{Hom}_{\mathcal{C}}(Q, -)\colon \mathcal{C} \to A$-mod and identifying $\mathcal{B}$ to B-mod by means of the equivalence $\mathrm{Hom}_{\mathcal{B}}(P, -)\colon \mathcal{B} \to B$-mod, Ψ identifies with ι^*, while $\Psi_!$ identifies with $\iota_!$.*

Proof. Let $\lambda \in \Lambda^+$. Applying $\Psi_!$ to the natural surjection $P(\mathcal{B}, \lambda) \to \Delta(\mathcal{B}, \lambda)$ yields a surjective map $\Psi_! P(\mathcal{B}, \lambda) \to \Psi_!\Delta(\mathcal{B}, \lambda)$. But $\Psi_!\Delta(\mathcal{B}, \lambda) \cong \Delta(\mathcal{C}, \lambda)$, while $\Psi_! P(\mathcal{C}, \lambda)$ is a projective object in $\mathcal{C}$ (because $\Psi_!$ has an exact right adjoint Ψ). Therefore, $Q = \Psi_! P$ maps surjectively onto $\Delta(\mathcal{C}, \lambda)$ and hence also maps onto $L(\mathcal{C}, \lambda)$. It follows that $P(\mathcal{C}, \lambda)$ is a direct summand of Q, for all $\lambda \in \Lambda^+$, and so Q is a projective generator for $\mathcal{C}$. This proves (1).

By Morita theory, $\mathrm{Hom}_{\mathcal{C}}(Q, -)\colon \mathcal{C} \to A$-mod and $\mathrm{Hom}_{\mathcal{B}}(P, -)\colon \mathcal{B} \to B$-mod are equivalences of categories, so that statement [CPS1, (4.6)] implies (2).

Now (3) is a direct verification: The equivalence $\mathcal{C} \xrightarrow{\sim} A$-mod is given by $M \mapsto \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$ for $M \in \mathrm{Ob}(\mathcal{C})$. Similarly, $N \mapsto \mathrm{Hom}_{\mathcal{B}}(P, N)$, $N \in \mathrm{Ob}(\mathcal{B})$, defines an equivalence $\mathcal{B} \xrightarrow{\sim} B$-mod. Therefore, $\mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M) \mapsto \mathrm{Hom}_{\mathcal{B}}(P, \Psi M)$, $M \in \mathrm{Ob}(\mathcal{C})$, defines an exact functor $j^*\colon A\text{-mod} \to B\text{-mod}$. We claim that if $\mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$ is regarded as a B-module using the homomorphism $\iota\colon B \to A$, then the isomorphism $\varphi_M\colon \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(P, \Psi M)$ is a B-module isomorphism. For $M \in \mathrm{Ob}(\mathcal{C})$ and $N \in \mathrm{Ob}(\mathcal{B})$, let $\tau_M\colon \Psi_!\Psi M \to M$ and

$\sigma_N\colon N \to \Psi\Psi_! N$ be the adjunction maps. Then φ_M is given by $\varphi_M(f) = \Psi(f)\sigma_P$ for $f \in \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$. Also, φ_M has inverse

$$\Psi_M\colon \mathrm{Hom}_{\mathcal{B}}(P, \Psi M) \to \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M), \quad g \mapsto \tau_M \Psi_!(g).$$

To show, for $f \in \mathrm{Hom}_{\mathcal{C}}(\Psi_! P, M)$, that $\varphi_M(f\Psi_!(b)) = \varphi_M(f)b$, for all $b \in B$, it suffices to check that

$$\begin{array}{ccc} P & \xrightarrow{\;b\;} & P \\ {\scriptstyle \sigma_P}\downarrow & & \downarrow{\scriptstyle \sigma_P} \\ \Psi\Psi_! P & \xrightarrow[\Psi\Psi_!(b)]{} & \Psi\Psi_! P \end{array}$$

is commutative. Since $\tau_{\Psi_! P}\Psi_!(\sigma_P) = \mathrm{id}$,

$$\begin{aligned} \sigma_P b &= \varphi_{\Psi_! P}\Psi_{\Psi_! P}(\sigma_P b) = \varphi_{\Psi_! P}(\tau_{\Psi_! P}\Psi_!(\sigma_P)\Psi_!(b)) \\ &= \varphi_{\Psi_! P}(\Psi_!(b)) = \Psi\Psi_!(b)\sigma_P, \end{aligned}$$

as desired. Thus $j^* \cong \iota^*$, i.e., j^* is induced by pull-back through $\iota\colon B \to A$. The other assertions of (3) are clear. □

In practice, the morphism $\iota\colon B \to A$ defined in (2.6) need not be an injection. Assume that (1.2) holds for $\mathcal{C}$. Then, as discussed in (2.3(3)), we can assume that $\leqslant = \leqslant'$. Then, using (2.3(4)), replace B by its image $B' = \iota(B)$ in A and replace Λ by the subposet indexing the irreducible B'-modules. Thus, A-mod $\to$ B'-mod defines a Borel category, realized as the module category of a subalgebra of A. For this reason, the remainder of this section focuses on subalgebras, though more general prealgebras could easily be considered. Throughout the discussion, A will be a quasi-hereditary algebra (with poset $(\Lambda^+, \leqslant^+)$). Usually, B will be a quasi-hereditary subalgebra (with poset $(\Lambda, \leqslant)$ – which thus determines its standard and costandard objects $\Delta(A, \lambda)$ and $\nabla(A, \lambda)$).

(2.7) DEFINITION. Let A be a quasi-hereditary algebra, with weight poset $(\Lambda^+, \leqslant^+)$. Let B be a quasi-hereditary subalgebra, with weight poset $(\Lambda, \leqslant)$, of A, and let $\Psi = |_B\colon A$-mod $\to B$-mod be the natural restriction (or pull-back) functor induced by the inclusion map $\iota\colon B \to A$. Then we say that B is a *Borel subalgebra* of A provided Ψ defines B-mod as a Borel category associated to A-mod.⋆

In the case $\Psi\colon A$-mod $\to B$-mod is the restriction functor, Condition (B0) in (2.2) is automatic: the tensor induction $A \otimes_B -$ provides a left adjoint to Ψ.

⋆ Since the opposite algebra A^{op} is also a quasi-hereditary algebra (with the same weight poset $(\Lambda^+, \leqslant^+)$), a subalgebra B^- of A such that $(B^-)^{\mathrm{op}}$ is a Borel subalgebra of A^{op} might be called an 'opposite Borel subalgebra' of A. Sometimes it is useful to consider both notions together, cf. (2.11), (2.14), (3.7) and (6.12).

Therefore, to verify whether B is a Borel subalgebra of A, we need only check Conditions (B1)–(B3) in (2.2) with $\mathcal{C} = A$-mod and $\mathcal{B} = B$-mod.

If $\preceq$ is a poset structure on the weight set Λ^+ of A, a $\preceq$-*adapted listing* of Λ^+ consists of a listing $\lambda_1, \ldots, \lambda_n$ of the set Λ^+ such that $\lambda_i \preceq \lambda_j$ implies $i \geqslant j$. In this case, there exists a defining sequence

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A \tag{2.8}$$

of idempotent ideals of A (as per [CPS1, §3]) such that the A/J_{i-1}-module J_i/J_{i-1} is a direct sum of copies of $\Delta(A, \lambda_i)$ (and is a projective left or right A/J_{i-1}-module). We call (2.8) a $\preceq$-*adapted* defining sequence of A. In case B is a Borel subalgebra of A, we say that (2.8) is $\leqslant$-adapted provided it is $\leqslant|_{\Lambda^+}$-adapted. (Here $(\Lambda, \leqslant)$ is the weight poset for B-mod in (2.7).) Because $\leqslant^+$ is compatible with $\leqslant$, any $\leqslant$-adapted defined sequence is also $\leqslant^+$-adapted.

With this terminology, we can establish the following result which provides some necessary and sufficient conditions for a subalgebra B of A to be a Borel subalgebra.

(2.9) THEOREM. *Let A be a quasi-hereditary algebra as above.*

(1) *Suppose B is a Borel subalgebra of A as per* (2.7). *Let $\{\lambda_1, \ldots, \lambda_n\}$ be a $\leqslant$-adapted listing of Λ^+, and let $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$ be the corresponding $\leqslant$-adapted defining sequence of A. For each i, let e_i be a primitive idempotent of B satisfying $e_i L(B, \lambda_i) \neq 0$. Then $e_i \in J_i$, $e_i \notin J_{i-1}$, and the following equalities hold:*

$$e_i B + J_{i-1} = e_i A + J_{i-1}, \tag{2.9.1}$$

$$e_i B e_i + J_{i-1} = e_i A e_i + J_{i-1}. \tag{2.9.2}$$

(2) *Assume also that* (1.2) *holds for A. Then there is a B-module isomorphism*

$$L(B, \lambda_i) \cong Be_i/(Be_i \cap J_{i-1}) \big(\cong (Be_i + J_{i-1})/J_{i-1}\big). \tag{2.9.3}$$

(3) *Conversely, let B be a subalgebra of A directed by a partial ordering $\leqslant'$ on its weight set Λ. Let $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n = A$ be a defining sequence for A such that each J_i/J_{i-1} is a direct sum of copies of $\Delta(A, \lambda_i)$. Suppose there exist primitive idempotents $e_1, \ldots, e_n \in B$ such that $e_i \in J_i \backslash J_{i-1}$ and* (2.9.1) *holds for each i. Assume A satisfies* (1.2). *Then B* (*with poset* $(\Lambda, \leqslant')$) *is a Borel subalgebra of A* (*with poset* $(\Lambda^+, \leqslant^+_{\min})$).

Proof. We first prove (1). Fix i and let $e_i \in B$ be a primitive idempotent such that $e_i L(B, \lambda_i) \neq 0$. By $(*)$ in the proof of (2.1), $[\Psi L(A, \lambda_i) : L(B, \nu)] \neq 0 \Rightarrow \nu \leqslant \lambda_i$. Also, $L(B, \lambda_i)$ is a multiplicity 1 composition factor of $\Psi L(A, \lambda_i)$, and it occurs in the socle. Thus, the isomorphism $\mathrm{Hom}_A(\Psi_! Be_i, L(A, \lambda_j)) \cong \mathrm{Hom}_B(Be_i, \Psi L(A, \lambda_j))$ shows that the head of $Ae_i = \Psi_! Be_i$ contains $L(A, \lambda_i)$ with multiplicity one, together with summands $L(A, \lambda_j)$ with $j < i$. Thus,

$(A/J_i)e_i = 0$. In particular, $e_i \in J_i$. Also, $(A/J_{i-1})e_i$ has a simple head $L(A, \lambda_i)$, so the image of e_i in A/J_{i-1} is a primitive idempotent (thus, $e_i \notin J_{i-1}$).

It follows that $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)$, while $e_i(A/J_{i-1}) \cong \nabla(A, \lambda_i)^*$ (K-linear dual). By property (B2), the restriction $\Psi\nabla(A, \lambda_i) = \nabla(A, \lambda_i)|_B$ is isomorphic to a B-submodule of $\nabla(B, \lambda_i)$, and hence to a B-submodule of $I(B, \lambda_i) \cong (e_iB)^*$. So, $e_i(A/J_{i-1})$ is a right B-module homomorphic image of e_iB, and hence of

$$e_iB/(e_iB \cap J_{i-1}) \cong e_i((B + J_{i-1})/J_{i-1}).$$

However, $e_i((B + J_{i-1})/J_{i-1})$ is a B-submodule of $e_i(A/J_{i-1})$. It follows that $e_i(A/J_{i-1}) \cong e_iB/(e_iB \cap J_{i-1})$. Thus, (2.9.1) must hold, and (2.9.2) follows by multiplying (2.9.1) on the right by e_i. This proves (1).

To prove (2), let $\leqslant'$ be the partial ordering on Λ directing B-mod. Being a quotient module of $Be_i \cong P(B, \lambda_i)$, B-module $Be_i/(Be_i \cap J_{i-1})$ has head $L(B, \lambda_i) \cong \Delta(B, \lambda_i)$, and the other composition factors $L(B, \nu)$ of $Be_i/(Be_i \cap J_{i-1})$ satisfy $\nu >' \lambda_i$. Hence, if (2.9.3) does not hold, $Be_i/(Be_i \cap J_{i-1})$ must have some $L(B, \nu)$ with $\nu >' \lambda_i$ in its socle. Since $Be_i/(Be_i \cap J_{i-1})$ is isomorphic to a B-submodule of $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)$, we obtain a nonzero B-module homomorphism $\Delta(B, \leqslant, \nu) \to \Psi\Delta(A, \lambda_i)$, hence a nonzero A-module homomorphism $\Psi_!\Delta(B, \leqslant, \nu) \to \Delta(A, \lambda_i)$, by adjointness. In particular, $\Psi_!\Delta(B, \leqslant, \nu) \neq 0$. It follows that $\nu \in \Lambda^+$ and $\Psi_!\Delta(B, \leqslant, \nu) \cong \Delta(A, \nu)$, by (2.1). Therefore, $\nu \leqslant^+_{\min} \lambda_i$. Since (1.2) holds for A, $\leqslant^+_{\min}$ is compatible with $\leqslant'$, see (2.3(3)). Thus, $\nu \leqslant' \lambda_i$, a contradiction, which shows that (2.9.3) must hold. This completes the proof of (2).

Assuming the hypotheses of (3), we verify conditions (B1)–(B3) in (2.2).

Because each $e_i \in B$ is primitive and B is directed, the algebra $e_iBe_i \cong \mathsf{K}$. Thus, (2.9.2) implies that $e_i(A/J_{i-1})e_i \cong e_iBe_i \cong \mathsf{K}$, so the image of e_i in A/J_{i-1} must also be primitive. Let $L(A, \lambda_i)$ be the corresponding irreducible A-module, i.e., $L(A, \lambda_i)$ is isomorphic to the head of $(A/J_{i-1})e_i$. We let $\lambda_i \in \Lambda$ also index the irreducible B-module corresponding to e_i, i.e., $e_iL(B, \lambda_i) \neq 0$. In this way, we regard Λ^+ as a subset of Λ. (Observe that the e_i are pair-wise not equivalent, since the J_i are distinct.) Now condition (B3) holds, since $\leqslant'$ directs B.

Since $e_i \in J_i$, the module $e_i(A/J_{i-1}) = e_i(J_i/J_{i-1})$ is an indecomposable summand of the right A-module J_i/J_{i-1}. Thus, $e_i(A/J_{i-1}) \cong \nabla(A, \lambda_i)^*$. It follows from (2.9.1) again that $\Psi\nabla(A, \lambda_i)$ is a (nonzero) B-submodule of $(e_iB)^* \cong I(B, \lambda_i) = \nabla(B, \lambda_i)$. So (B2) holds.

It remains to prove, under the assumption of (1.2), that $\leqslant^+_{\min}$ is compatible with $\leqslant'$. But if $L(A, \mu)$ is a composition factor of $\nabla(A, \lambda)$, then, by $(*)$ in the proof of (2.1), the socle $L(B, \mu)$ of $\Psi L(A, \mu)$ is a composition factor of $\Psi\nabla(A, \lambda)$, and hence of $\nabla(B, \lambda)$. Thus, $\mu \leqslant' \lambda$, as required. □

(2.10) *Remarks*. (1) From the proof of (2.9(3)), one can see that the assumption that A satisfies (1.2) and the usage of the partial ordering $\leqslant^+_{\min}$ in (2.9(3)) are only necessary for the compatibility of the partial ordering on Λ^+ with that on Λ. In practice, it is often the case that this compatibility is automatic. In this case, we can use (2.9(3)) without (1.2) and $\leqslant^+_{\min}$. See (2.13) and (3.5).

(2) Suppose that in (2.9), it is assumed that the directed algebra B arises as a path algebra. (It is always naturally the image of a finite-dimensional path algebra.) Then the ordering $\leqslant_{\min}$ on Λ may be described completely by the projective indecomposable B-modules. That is, $\lambda_i \leqslant \lambda_j$ if and only if $e_j B e_i \neq 0$. In particular, if $\leqslant^+_{\min}$ is compatible with $\leqslant_{\min}$, then

$$e_i A e_j \not\subseteq J_{j-1} \Rightarrow e_j B e_i \neq 0 \tag{2.10.1}$$

holds for all i, j. Conversely, in this case, part (3) remains true if the condition that (1.2) holds is replaced by condition (2.10.1). We leave the details to the reader. Finally, we remark that (2.9) can be reformulated in terms of prealgebras. In that case, we could assume that B is a path algebra. (Alternatively, even in the subalgebra case, one could replace the right-hand side of (2.10.1) by the more complicated "path" condition: There exist $i_1, \ldots, i_n$ with $i_1 = j$, $i_n = i$ and $e_i B e_{i+1} \neq 0$ for all $i < n$.)

The following immediate corollary uses the fact that, if A is a quasi-hereditary algebra with weight poset $(\Lambda^+, \leqslant^+)$, then the opposite algebra A^{op} is also a quasi-hereditary algebra with weight poset $(\Lambda^+, \leqslant^+)$. If $\{J_\bullet\}$ is a $\leqslant^+$-adapted defining sequence for A, it remains a $\leqslant^+$-adapted defining sequence for A^{op}. The corollary is a generalization of a factorization for Schur algebras by [G] and for q-Schur algebras by [PW]. See (6.11).

(2.11) COROLLARY. *Let A be a quasi-hereditary algebra. Suppose B and B^- are quasi-hereditary subalgebras of A which have the same weight poset $(\Lambda, \leqslant)$. Assume that B is a Borel subalgebra of A and that $(B^-)^{\mathrm{op}}$ is a Borel subalgebra of A^{op}. Assume that $\{J_\bullet\}$ is a $\leqslant$-adapted defining sequence of A, and that there exist idempotents $e_i \in B \cap B^-$ which are primitive in both B and B^- and satisfy $e_i L(B, \lambda_i) \neq 0$, $e_i L((B^-)^{\mathrm{op}}, \lambda_i) \neq 0$.*

Then there is a "triangular factorization" $A = B^- B$. More precisely, $A = \bigcup_i B^- e_i B$. If $\varphi_i\colon\ J_i \to J_i/J_{i-1}$ is the natural quotient map, then $J_i/J_{i-1} \cong \varphi_i(B^- e_i) \otimes_K \varphi_i(e_i B)$.

(2.12) *Remark.* The above corollary is inspired by a similar result for Schur algebras proved by Green [G]. In turn, Green's result was extended to q-Schur algebras in [PW, (11.6.1)] – see Section 6 below. Also, [DR, (3.5), (5.5)] obtains such a factorization for q-Schur^2-algebras. These algebras do not obviously arise from Lie theory, but are interesting for the representation theory of finite groups of Lie type in the non-describing characteristics. Finally, [Dy] obtains a suggestive and similar factorization, in an entirely different context. (See Section 3.)

The next result shows that every quasi-hereditary algebra A has at least one Borel subalgebra B. It can happen that $A = B$ (e.g., when A is directed by $\leqslant^+$) or that $B =$ (e.g., when A is directed by the partial ordering opposite to $\leqslant^+$). The next section shows that, under suitable assumptions, nice properties of A imply that a Borel subalgebra B (usually different than below) can be chosen to have similar nice properties. Below, we find that the irreducible A-modules restrict to irreducible B-modules. Such Borel subalgebras are called *strong* – see Section 3.

(2.13) THEOREM. *A quasi-hereditary algebra A has a Borel subalgebra B satisfying:*

(i) *it has weight poset $(\Lambda, \leqslant)$ in which $\Lambda = \Lambda^+$;*
(ii) $\leqslant$ *directs B-*mod*; and*
(iii) *every irreducible A-module restricts to an irreducible B-module.*

Proof. Replacing $\leqslant^+$ with $\leqslant^+_{\min}$, we assume that $\leqslant^+ = \leqslant^+_{\min}$. First, we consider the case A is a basic algebra. Let $\lambda_1, \dots, \lambda_n$ be a $\leqslant^+$-adapted listing of Λ^+, and form the associated $\leqslant^+$-adapted defining sequence (2.8) for A.

Let A_0 be a Wedderburn complement for $\mathrm{rad}(A)$ in A, and let $e_1, \dots, e_n \in A_0$ be a complete set of primitive orthogonal idempotents, listed so that $\dim e_i L(A, \lambda_j) = \delta_{ij}$ for all i, j. Since A is basic, the e_i form a basis for the commutative algebra A_0. If $f_i = e_1 + \cdots + e_i$, then $J_i = Af_iA$.

The subalgebra $B = A_0 \oplus \sum_{i<j} e_i \mathrm{rad}(A) e_j$ has weight set $\Lambda = \Lambda^+$. Since

$$\mathrm{Hom}_B(Be_i, Be_j) \cong e_i B e_j \cong \begin{cases} e_i \mathrm{rad}(A) e_j, & i < j, \\ \mathsf{K}, & i = j, \\ 0, & i > j, \end{cases}$$

the composition factors $L(B, \lambda_i)$ of the radical of Be_j satisfy $i < j$. It is easy to define a partial ordering $\leqslant$ on Λ such that $\leqslant$ is compatible with $\leqslant^+$, and that $\leqslant$ directs B-mod. For example, define $\lambda_i \leqslant \lambda_j$ if and only if $i \geqslant j$, or define $\leqslant$ to be generated by the preorder $\mu \leqslant_{\mathrm{pre}} \lambda$ if and only if $L(B, \lambda)$ is a composition factor of $P(B, \mu)$. Fix such an ordering $\leqslant$. We claim that B (with weight poset $(\Lambda, \leqslant)$) is a Borel subalgebra of A (with weight poset $(\Lambda, \leqslant^+)$). By (2.9(3)) (now (1.2) is unnecessary, see Remark (2.10(1))), it suffices to verify that the equalities (2.9.1) hold for all i.

For each i, $(A/J_{i-1})e_i \cong \Delta(A, \lambda_i)$, so that

$$e_i(A/J_{i-1})e_i \cong \mathrm{Hom}_A(Ae_i, \Delta(A, \lambda_i)) \cong \mathsf{K}.$$

Hence,

$$e_i A e_i \subseteq \mathsf{K} e_i + J_{i-1}. \tag{2.13.1}$$

Since $e_i B = \mathsf{K} e_i + \sum_{j>i} e_i \mathrm{rad}(A) e_j$, and $e_j \in J_{i-1}$ for $j < i$,

$$\begin{aligned} e_i A &= \mathsf{K} e_i + \sum_j e_i \mathrm{rad}(A) e_j \\ &= \mathsf{K} e_i + \sum_{j>i} e_i \mathrm{rad}(A) e_j + \sum_{j \leqslant i} e_i \mathrm{rad}(A) e_j \\ &\equiv e_i B \mod J_{i-1}, \end{aligned}$$

by (2.13.1). So (2.9.1) holds. Hence, B is a Borel subalgebra of A. By construction, the irreducible A-modules restrict to irreducible B-modules. Thus, the theorem holds when A is basic.

An algebra Morita equivalent to the basic algebra A has the form

$$A' = \mathrm{End}_A\Big(\bigoplus (Ae_i)^{\oplus m_i}\Big)^{\mathrm{op}}$$

for some sequence $m_1, \ldots, m_n$ of *positive* integers. Then

$$B' = \mathrm{End}_B\Big(\bigoplus (Be_i)^{\oplus m_i}\Big)^{\mathrm{op}}$$

is a subalgebra of A' which is Morita equivalent to B. Thus, B' defines a Borel subalgebra of A'. The proof is complete. □

As a corollary to (2.13) and its proof, we have the following result.

(2.14) COROLLARY. *Every quasi-hereditary algebra A has subalgebras B and B^- satisfying the hypothesis of* (2.11), *and thus has a corresponding 'triangular factorization' $A = B^- B$.*

Proof. In the notation of (2.13), if $B^- = A_0 \oplus \sum_{i>j} e_i \mathrm{rad}(A) e_j$, then $(B^-)^{\mathrm{op}}$ is a Borel subalgebra of A^{op}. Both B-mod and $(B^-)^{\mathrm{op}}$-mod are directed by defining $\lambda_i \leqslant \lambda_j$ if and only if $i \geqslant j$. Then $\leqslant^+$ is compatible with this poset structure. Trivially, the hypotheses of (2.11) hold. □

The following proposition was inspired by (2.9), but can be proved directly using (2.5).

(2.15) PROPOSITION. *Suppose B is a Borel subalgebra* (*with weight poset* $(\Lambda, \leqslant)$) *of a quasi-hereditary algebra A* (*with weight poset* $(\Lambda^+, \leqslant^+)$). *Assume that $\leqslant$ directs B*-mod. *Then:*

(1) *Let $e \in B$ be an idempotent such that $eL(B, \lambda) \neq 0$ for each $\lambda \in \Lambda^+$. Then eAe is Morita equivalent to A, and eBe is a Borel subalgebra of eAe.*
(2) *There exists an idempotent $e \in B$ such that eBe is a Borel subalgebra* (*with weight poset* $(\Lambda^+, \leqslant|_{\Lambda^+})$ *of eAe* (*with weight poset* $(\Lambda^+, \leqslant^+)$). *We can even assume that eBe is basic.*

Proof. (1) Clearly, eAe-mod $\cong A$-mod. Let $\Gamma = \{\mu \in \Lambda \mid eL(\mathcal{B}, \mu) = 0\}$, and $\mathcal{S}$ be the Serre subcategory of $\mathcal{B}$ generated by $L(\mathcal{B}, \mu)$ for $\mu \in \Gamma$. Then $\mathcal{B}/\mathcal{S} \cong eBe$-mod. Thus, (2.5) applies to the present situation, yielding a Borel category eAemod $\to eBe$-mod. Since eBe is a subalgebra of eAe, (1) is proved. By construction, eBe-mod has weight poset $(\Lambda \backslash \Gamma, \leqslant|_{\Lambda\backslash\Gamma})$.

(2) This is a special case of (1), provided $e \in B$ also satisfies $eL(\mathcal{B}, \lambda) = 0$ for all $\lambda \notin \Lambda^+$. We may choose such an e so that eBe is basic. □

A Borel category or subalgebra has *only dominant weights* provided $\Lambda = \Lambda^+$ as sets. This condition is necessary and sufficient for $\Psi_!$ to kill only zero objects. (Note that Ψ never kills any nonzero object or map. See (2.3(1)).) It can always be achieved by suitably choosing the idempotent e above. This provides a replacement for the flawed [K1, Appendix, Thm. F]. The construction in (2.15b) above is essentially the same as in this latter reference. If B is the Borel subalgebra of the Schur algebra $S(n, r)$ due to Green (see (6.11) below with $q = 1$), then the construction eBe leads generally to a Borel subalgebra of $eS(n, r)e$ which is too big to be "exact" in the sense of Section 4. (The condition (4.1(1)) below fails in general.) This observation is due to S. König (personal communication), e.g., (4.1(1)) fails for $S(3, 5)$ in characteristic 0 by a straightforward weight argument.

3. Borel Subalgebras and Kazhdan–Lusztig Theory

The construction of Borel subalgebras given in (2.9) applies to an important class of quasi-hereditary algebras which arise in the representation theory of algebraic groups, quantum groups, etc. The main result, given in (3.5), is inspired by a similar (but slightly weaker) unpublished result of the first two authors and E. Cline – see Corollary (3.6). Another application yields a strengthening in (3.7) of a result of Dyer [Dy].

We begin with some preliminaries on graded algebras and graded modules.

Let $A = \bigoplus_{n\geqslant 0} A_n$ be a positively $\mathbb{Z}$-graded (finite-dimensional) algebra over K, and assume that A_0 is semisimple. (In what follows, we will refer to such algebras simply as 'graded' algebras.) Let $\mathcal{C}^{\rm gr}$ be the category of finitely generated $\mathbb{Z}$-graded A-modules. If $M = \bigoplus_n M_n \in \mathrm{Ob}(\mathcal{C}^{\rm gr})$, then, for any integer i, the "twist" $M(i)$ is the graded A-module obtained from M by shifting the grading i steps to the right, e.g., $M(i)_n = M_{n-i}$. Let Λ^+ be the set of weights of A-mod. Since $\mathrm{rad}(A) = \bigoplus_{n>0} A_n$, we can regard any irreducible A-module $L(\lambda)$, $\lambda \in \Lambda^+$, as a graded A-module, concentrated in degree 0. Then the graded modules $L(\lambda)(i)$, $\lambda \in \Lambda^+$, $i \in \mathbb{Z}$, are representatives from the set of distinct isomorphism classes of irreducible objects in $\mathcal{C}^{\rm gr}$.

The reader should keep in mind that, given $M, N \in \mathrm{Ob}(\mathcal{C}^{\rm gr})$, we have

$$\mathrm{Ext}^\bullet_{\mathcal{C}}(M, N) \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^\bullet_{\mathcal{C}^{\rm gr}}(M, N(i)), \tag{3.1}$$

relating the 'ungraded $\mathrm{Ext}^\bullet$-groups' with the 'graded $\mathrm{Ext}^\bullet$-groups'. We sometimes write $\mathrm{Ext}^\bullet_{A^{\rm gr}}$ in place of $\mathrm{Ext}^\bullet_{\mathcal{C}^{\rm gr}}$.

The graded algebra A is called *tightly graded* provided that A is isomorphic to the graded algebra $\mathrm{gr}\,A = \bigoplus_i \mathrm{rad}(A)^i/\mathrm{rad}(A)^{i+1}$ obtained from the radical filtration of A. Equivalently, A_0 is semisimple and A is generated as an A_0-module by its term A_1 in grade 1. The following result summarizes some elementary properties of tightly graded algebras.

(3.2) LEMMA. *Let A be a tightly graded algebra. Let L, L' be irreducible A-modules* (*viewed as graded A-modules concentrated in degree* 0). *Then:*

(1) *If $m, n \in \mathbb{Z}$ are integers such that* $\mathrm{Ext}^1_{A^{\mathrm{gr}}}(L'(m), L(n)) \neq 0$, *then $m + 1 = n$.*
(2) *If $e, e' \in A_0$ are primitive idempotents such that $eL \neq 0$, $e'L \neq 0$, then*

$$\mathrm{Ext}^1_A(L', L) \cong \mathrm{Ext}^1_{A^{\mathrm{gr}}}(L', L(1)) \cong eA_1e'$$

as vector spaces.

Proof. (1) After twisting, we assume that $m = 0$. Let $0 \to L(n) \to E \to L' \to 0$ be a nonsplit graded extension of L' by $L(n)$. Take $0 \neq v \in E_0$ which maps to a nonzero element in $L' = L'_0$. Then $E = Av = A_0v \oplus A_1v \oplus \cdots$. Since A_1 generates A as an A_0 module, $A_1v \neq 0$ and so the socle $L(n)$ of E must have degree $n = 1$.

(2) Let Q' be the radical of Ae', i.e., $Q' = \bigoplus_{n>0} A_ne'$. By (1) and (3.1),

$$\begin{aligned}\mathrm{Ext}^1_A(L', L) &\cong \mathrm{Ext}^1_{A^{\mathrm{gr}}}(L', L(1))\\ &\cong \mathrm{Hom}_{A^{\mathrm{gr}}}(Q', L(1)) \cong \mathrm{Hom}_{A_0}(A_1e', L) \cong eA_1e',\end{aligned}$$

as required. □

The proof of the main result below will be based on the following hypothesis concerning Ext^1_A for a quasi-hereditary algebra A. (In this hypothesis, there is no assumption that A is graded.)

(3.3) HYPOTHESIS. *Let $\mathcal{C} = A$-mod be a highest weight category with weight poset* $(\Lambda^+, \leqslant^+)$. *For any $\lambda, \mu \in \Lambda^+$, the 'restriction map'* (*induced by the surjection* $\Delta(\lambda) \twoheadrightarrow L(\lambda)$)

$$\mathrm{Ext}^1_A(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_A(\Delta(\lambda), L(\mu)) \quad \textit{is surjective}. \tag{3.3.1}$$

(3.4) *Remark.* In order to motivate the above hypothesis we recall some results from [CPS3, CPS4, CPS6, CPS7]. Again, let $\mathcal{C}$ be a highest weight category with weight poset Λ^+. Let $\ell\colon \Lambda^+ \to \mathbb{Z}$ be a function (the 'length function' on weights). We say that $\mathcal{C}$ has a Kazhdan–Lusztig theory (relative to ℓ) provided that, given $\lambda, \mu \in \Lambda^+$, the nonvanishing of either $\mathrm{Ext}^n_{\mathcal{C}}(\Delta(\lambda), L(\mu))$ or $\mathrm{Ext}^n_{\mathcal{C}}(L(\lambda), \nabla(\mu))$ implies that $\ell(\lambda) - \ell(\mu) \equiv n$ (mod 2).

In case $\mathcal{C}$ has a Kazhdan–Lusztig theory, the surjectivity of the maps (3.3.1) holds for all weights λ, μ [CPS4, (4.3)]. (An analogue even holds for all n.) In all the standard examples arising in the representation theory of algebraic groups, quantum groups, and the category $\mathcal{O}$, the surjectivity condition (3.3.1) essentially *implies* the existence of a Kazhdan–Lusztig theory; see [CPS4]. This fact will hold generally in situations in which a suitably rich supply of 'Hecke operators' exists.

Now suppose that $\mathcal{C} = A$-mod and that the algebra A is graded as above. Then $\mathcal{C}^{\mathrm{gr}}$ is a *graded* highest weight category in the sense of [CPS3]. In particular, the standard objects $\Delta(\lambda)$ and the costandard objects $\nabla(\lambda)$ have a natural graded structure (with the head of $\Delta(\lambda)$ and the socle of $\nabla(\lambda)$ having grade 0). In this graded setting, it is also possible to consider parity conditions along the lines of those expressed by a Kazhdan–Lusztig theory for $\mathcal{C}$. Namely, we say $\mathcal{C}^{\mathrm{gr}}$ has a *graded* Kazhdan–Lusztig theory relative to the length function $\ell\colon\ \Lambda \to \mathbb{Z}$ provided the nonvanishing of either $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{gr}}}(\Delta(\nu), L(\lambda)(m))$ or $\mathrm{Ext}^n_{\mathcal{C}^{\mathrm{gr}}}(L(\lambda), \nabla(\nu)(m))$ implies that $m = n \equiv \ell(\lambda) - \ell(\nu) \bmod 2$. From (3.1), it follows that if $\mathcal{C}^{\mathrm{gr}}$ has a graded Kazhdan–Lusztig theory, then the ungraded category $\mathcal{C}$ has a Kazhdan–Lusztig theory. Conversely, [CPS6, (3.9)] proves that A-mod has a graded Kazhdan–Lusztig theory if and only if it has a Kazhdan–Lusztig theory and A is a Koszul algebra.⋆

In (2.13), we proved the existence of a Borel subalgebra B of a quasi-hereditary algebra A. The Borel subalgebra constructed there has the same weight set as A. In other words, the Borel algebra "has dominant weights only". The following result establishs the graded version of (2.13). That is, we consider the existence of a tightly graded Borel subalgebra "having dominant weights only" for a tightly graded quasi-hereditary algebra A. The motivation for the construction of B below (as well as in (2.13)) comes from [Dy]. The importance of the 'quiver condition' (3.5.1) below was first noted by König in a related context, cf. [K2, K3].

(3.5) THEOREM. *Let A be a tightly graded quasi-hereditary algebra, with weight poset $(\Lambda^+, \leqslant^+)$, satisfying* (3.3)*. Then A has a tightly graded Borel subalgebra B, with weight poset $(\Lambda, \leqslant) = (\Lambda^+, \leqslant^+)$, such that $\leqslant$ directs B*-mod*. In addition, we can require that the 'quiver condition'*

$$\mathrm{Ext}^1_B(L(B,\mu), L(B,\lambda)) \cong \mathrm{Ext}^1_A(L(A,\mu), L(A,\lambda)) \tag{3.5.1}$$

holds for all $\mu <^+ \lambda$ in Λ^+.⋆⋆

Proof. First, assume that A is basic. Let $\{e_\lambda\}$ be a complete set of primitive orthogonal idempotents in A, indexed by the set Λ^+. Assume that each $e_\lambda \in A_0$, and that $1 = \sum e_\lambda$ since A is basic. Define B to be the subalgebra of A generated by A_0 and the subspaces $B_1 = \bigoplus_{\mu <^+ \lambda} e_\lambda A_1 e_\mu$ of A_1. Then B is a tightly graded subalgebra of A.

The set Λ^+ also indexes the isomorphism classes of irreducible B-modules: the distinct irreducible A-modules restrict to give the distinct irreducible B-modules. By the construction of B and (3.2),

$$\mathrm{Ext}^1_B(L(B,\mu), L(B,\lambda))$$

⋆ A graded algebra A (with A_0 semisimple and all negative grades zero) is *Koszul* provided that for simple A_0-modules L, L', and $m, n, p \in \mathbb{Z}$,

$$\mathrm{Ext}^p_{A^{\mathrm{gr}}}(L(m), L'(n)) \neq 0 \Rightarrow n - m = p.$$

Koszul algebras are tightly graded.

⋆⋆ This means that the quiver of A agrees with the quiver of B in the ascending direction, i.e., if $\mu < \lambda$, then the number of arrows from node μ to node λ in the quiver of A equals the number of arrows from μ to λ in the quiver of B. In the case where A is basic and B exact, this is equivalent to König's condition $(*)$ in [K2, p. 604] asserting that A has a presentation by quivers and relations so that the vertices and arrows which are suitably ordered generate B.

$$\cong e_\lambda B_1 e_\mu$$
$$\cong \begin{cases} 0, & \mu \geqslant^+ \lambda, \\ e_\lambda A_1 e_\mu \cong \operatorname{Ext}^1_A(L(A,\mu), L(A,\lambda)), & \mu <^+ \lambda. \end{cases}$$

Thus, B-mod is directed by $\leqslant^+$ and (3.5.1) holds. Put $\leqslant = \leqslant^+$ on $\Lambda = \Lambda^+$.

Pick a $\leqslant^+$-adapted defining sequence $\{J_\bullet\}$, cf. (2.8). We will use (2.9(3)) (see also (2.10(1))) to show that B defines a Borel subalgebra of A. If the defining sequence corresponds to the listing $\lambda_1, \dots, \lambda_n$ of Λ^+, then write $e_i = e_{\lambda_i}$. Certainly, $e_i \in J_i \backslash J_{i-1}$. We will show that (2.9.1) holds.

Consider the case $i = 1$. We want to prove that

$$e_1 A \subseteq e_1 B. \tag{3.5.2}$$

Since $e_1 B \subseteq e_1 A$ automatically as right B-modules, we obtain (after taking linear duals) a surjection

$$(e_1 A)^* \cong \nabla(A, \lambda_1)|_B \twoheadrightarrow I(B, \lambda_1) \cong (e_1 B)^*. \tag{3.5.3}$$

Since $[\nabla(A,\lambda_1) : L(A,\lambda_1)] = 1$, we see that $[\nabla(A,\lambda_1)|_B : L(B,\lambda_1)] = 1$. Of course, $L(B,\lambda_1)$ occurs with multiplicity 1 in $I(B,\lambda_1)$ and is, in fact, the B-socle of $I(B,\lambda_1)$, so $L(B,\lambda_1)$ cannot be killed by the homomorphism (3.5.3). *If* $L(B,\lambda_1)$ *is* the B-socle of $\nabla(A,\lambda_1)$, then it follows that (3.5.3) is an isomorphism.

Since $\nabla(A,\lambda_1)$ is a graded A-module, it is a graded B-module. The B-socle of $\nabla(A,\lambda_1)$ is the B-submodule annihilated by the graded ideal $\operatorname{rad}(B)$, so it is graded, too.

Let $N \cong L(B,\mu)$ be in a homogeneous summand of the B-socle of $\nabla(A,\lambda_1)$ for some $\mu \neq \lambda_1$. Let $M = AN$ be the (graded) A-submodule of $\nabla(A,\lambda_1)$ generated by N. We will show that M has a graded quotient $\overline{M}$ which has precisely two composition factors: head $L(A,\mu)$ and socle $L(A,\nu)$ for some $\nu >^+ \mu$. The existence of such a quotient $\overline{M}$ will lead to a contradiction: (3.2) implies that the grade of $L(A,\nu)$ in $\overline{M}$ must be one bigger than the grade of $L(A,\mu)$. Hence, $e_\nu A_1 e_\mu \overline{M} \neq 0$. But $e_\nu A_1 e_\mu \subseteq \operatorname{rad}(B)$ and $\operatorname{rad}(B)N = 0$. This gives the contradiction, which shows that the B-socle of $\nabla(A,\lambda_1)$ is $L(A,\lambda_1)$, hence that (3.5.3) is an isomorphism.

Therefore, to prove (3.5.2), it suffices to find a quotient $\overline{M}$ with the above-mentioned property. We will establish this in the next two paragraphs.

Clearly, $L(A,\mu)$ is the head of M. Also, $L(A,\lambda_1)$ (where $\lambda_1 >^+ \mu$) is in the socle of M. So there exists a quotient module M' of M with exactly one composition factor of the form $L(A,\nu)$ with $\nu >^+ \mu$, and the factor appears in the socle. Let $M'' = M'/L(A,\nu)$. Then there exists a (graded) A-module morphism $\Delta(A,\mu) \to M''$ covering the image of N in M'' – hence, M'' is a homomorphic image of $\Delta(A,\mu)$. Pulling M'' back along the surjection $\Delta(A,\mu) \to M''$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(A,\nu) & \longrightarrow & E & \xrightarrow{f} & \Delta(A,\mu) & \longrightarrow & 0 \\ & & \downarrow \sim & & \downarrow g & & \downarrow \text{pull-back} & & \\ 0 & \longrightarrow & L(A,\nu) & \longrightarrow & M' & \longrightarrow & M'' & \longrightarrow & 0 \end{array} \tag{3.5.4}$$

By (3.3), the extension in the top row of (3.5.4) must be induced by an extension of the form $0 \to L(A,\nu) \to E' \to L(A,\mu) \to 0$. This implies that the exact sequence

$$0 \to L(A,\nu) \to f^{-1}\mathrm{rad}(\Delta(A,\mu)) \to \mathrm{rad}(\Delta(A,\mu)) \to 0$$

obtained from the top row splits in A-mod. So

$$f^{-1}\mathrm{rad}(\Delta(A,\mu)) \cong \mathrm{rad}(\Delta(A,\mu)) \oplus L(A,\nu).$$

Since $\mathrm{rad}(\Delta(A,\mu))$ and $L(A,\nu)$ has no common composition factors, $\mathrm{rad}(M') = g(f^{-1}\mathrm{rad}(\Delta(A,\mu))) \cong Z \oplus L(A,\nu)$, where $Z \cong \mathrm{rad}(M'')$. Thus, the graded A-module $\overline{M} = M'/Z$ is a homomorphic image of M with exactly two composition factors: $L(A,\mu)$ in its head and $L(A,\nu)$ in its socle.

As mentioned above, this proves that (2.9.1) holds for $i = 1$.

The algebra $\overline{A} = A/J_1$ is still tightly graded and satisfies (3.3) (by elementary recollement theory for highest weight categories). It contains the subalgebra $\overline{B} = B/(B \cap J_1)$ which is generated by $\overline{A}_0 = A_0/(A_0 \cap J_1)$ and the $e_\lambda \overline{A}_1 e_\mu$ for $\lambda > \mu$ in $\Lambda^+ \backslash \{\lambda_1\}$. The proof that (2.9.1) holds for all i now follows by induction on $\#\Lambda^+$.

Finally, the case of a non-basic algebra A' can be deduced from the basic case, just as at the end of the proof of (2.13). One needs only observe that if A is tightly graded basic algebra, then any algebra A' which is Morita equivalent to A inherits a natural tight grading. We leave further details to the interested reader. □

The following unpublished result of Cline, Parshall, and Scott follows from (3.4) and (3.5).

(3.6) COROLLARY. *Assume that* A-mod *has a graded Kazhdan–Lusztig theory relative to a length function* $\ell \colon \Lambda^+ \to \mathbb{Z}$. *Then* A *has a directed, tightly graded Borel subalgebra* B *satisfying the 'quiver condition'* (3.5.1).

The following result was proved under stronger hypotheses (essentially those of (3.6)) by Dyer [Dy].

(3.7) COROLLARY. *Assume that* A *is a tightly graded quasi-hereditary algebra satisfying* (3.2). *Then there exist tightly graded subalgebras* B *and* B^- *such that* B (*resp.*, $(B^-)^{\mathrm{op}}$) *is a Borel subalgebra of* A (*resp.*, A^{op}), *and that* $A = B^- B$.

Proof. We define B as in (3.5), and take B^- to be the subalgebra of A generated by A_0 and the subspaces $e_\lambda A_1 e_\mu$ for $\lambda <^+ \mu$ (in the notation of the proof of (3.6)). Then the proof of (2.11) applies to give the factorization $A = B^- B$. □

We will say that a Borel subalgebra B is *strong* if every irreducible A-module restricts to an irreducible B-module.⋆ In this case, B must have dominant weights only (i.e., $\Lambda^+ = \Lambda$), since, as with any subalgebra, every irreducible B-module must occur as a composition factor of some irreducible A-module. (This is not automatic in any prealgebra or categorical setting. There is a corresponding categorical *strong* concept, the requirement on a Borel category that $\Psi L(\mathcal{C}, \lambda)$ be irreducible for each irreducible object $L(\mathcal{C}, \lambda)$ in $\mathcal{C}$, together with the requirement that $\mathcal{B}$ have dominant weights only.)

The statement (2) in the following proposition forms a converse, in some sense, to (3.5).

(3.8) PROPOSITION. *Suppose A is a quasi-hereditary algebra with a strong Borel subalgebra B. Then:*

(1) *We have*

$$\begin{aligned} &\dim \operatorname{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)) \\ &\quad \leqslant \dim \operatorname{Ext}^1_B(L(B,\lambda), L(B,\mu)), \quad \forall \lambda, \mu \in \Lambda^+. \end{aligned} \tag{3.8.1}$$

(2) *If we assume, in addition, that the 'quiver condition'* (3.5.1) *is satisfied for all $\lambda <^+ \mu$, then the 'restriction map'*

$$\operatorname{Ext}^1_A(L(A,\lambda), L(A,\mu)) \to \operatorname{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)) \tag{3.8.2}$$

is an isomorphism for all $\lambda <^+ \mu$ in Λ^+. In particular, (3.3) *holds for the category A*-mod.

Proof. (1) For $\lambda \in \Lambda^+$, let

$$P_1 \to P_0 \to L(B,\lambda) \to 0 \tag{3.8.3}$$

be the first terms of a minimal projective resolution of $L(B,\lambda)$. Thus,

$$\dim \operatorname{Hom}_B(P_1, L(B,\mu)) = \dim \operatorname{Ext}^1_B(L(B,\lambda), L(B,\mu)).$$

Applying $\Psi_! = A \otimes_B -$ to (3.8.3) gives a presentation $\Psi_! P_1 \to \Psi_! P_0 \to \Delta(A,\lambda) \to 0$ of $\Delta(A,\lambda) = \Psi_! L(B,\lambda)$ by projective modules. Thus,

$$\dim \operatorname{Hom}_A(\Psi_! P_1, L(A,\mu)) \geqslant \dim \operatorname{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)).$$

Comparison with the equality above and using $\Psi L(A,\mu) \cong L(B,\mu)$ gives the result.

⋆ This definition is equivalent in the exact case (see Sections 4 and 5) to a similar concept introduced by König [K1].

(2) Assume $\lambda <^+ \mu$. By (3.8.1) and the 'quiver condition' (3.5.1),

$$\dim \operatorname{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)) \leqslant \dim \operatorname{Ext}^1_A(L(A,\lambda), L(A,\mu)). \tag{3.8.4}$$

On the other hand, consider the exact sequence $0 \to Q(A,\lambda) \to \Delta(A,\lambda) \to L(A,\lambda) \to 0$. Since $\operatorname{Hom}_A(Q(A,\lambda), L(A,\mu)) = 0$, we deduce from the cohomological long exact sequence (obtained by applying $\operatorname{Hom}_A(-, L(A,\mu))$ to the above exact sequence) that the 'restriction map' (3.8.2) is injective. Now (3.8.4) forces the surjectivity of (3.8.2). Taking into account the fact that $\operatorname{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)) = 0$ for $\lambda \not<^+ \mu$ we see that (3.3) holds. □

(3.9) *Remark* (Equivalence with the (Kazhdan–)Lusztig conjecture). In the representation theories of reductive groups over an algebraically closed field K of characteristic $p > 0$, quantum groups (or quantum enveloping algebras) at a root of unity, and in the study of the category $\mathcal{O}$ for a complex simple Lie algebra, there is a 'Lusztig' (or 'Kazhdan–Lusztig') conjecture predicting the characters of certain irreducible modules. For $\mathcal{O}$, the conjectured character formula has been proved in [BB, BK] (independently). For quantum enveloping algebras at a root of unity, the conjecture follows from the category equivalence proved in [KL], together with a corresponding character formula [KT] for irreducible representations of affine Lie algebras at a negative level. Using that work, [AJS] established the Lusztig conjecture for reductive groups provided that p is 'large enough' (depending on the root system) – though no sufficient bound on p is known in *any* nontrivial cases.

The results in [CPS4, CPS5, CPS7] establish the equivalence of these conjectures with various (essentially homological) properties for certain highest weight categories $\mathcal{C}$ (or associated quasi-hereditary algebras A). Fix such a $\mathcal{C} = A$-mod with weight poset Λ^+. Then the conjectured character formula is equivalent to the existence of a Kazhdan–Lusztig theory on $\mathcal{C}$, using an appropriate length function $\ell\colon \Lambda^+ \to \mathbb{Z}$. Furthermore, as already remarked, the existence of a Kazhdan–Lusztig theory itself is equivalent in these cases to the truth of (3.3) on the surjectivity of the 'restriction' map (3.3.1). Thus, if A has a strong Borel subalgebra B satisfying condition (3.5.1), (3.8(2)) implies that (3.3) holds, and the (Kazhdan–)Lusztig conjecture holds for $\mathcal{C}$.*

When A is Koszul**, the (Kazhdan–)Lusztig conjecture implies conversely that A has a strong Borel subalgebra: In this case, the conjectured character formula implies that $\mathcal{C}$ has a *graded* Kazhdan–Lusztig theory – cf. (3.4). Therefore, Corollary (3.6) implies that A has a strong Borel subalgebra B which is tightly graded

* König makes a very similar claim in his announcement [K2, p. 604] but with the additional requirement that B is exact. (No proof is included or has yet appeared.)

** The algebra A is Koszul when $\mathcal{C} = \mathcal{O}$. (See, e.g., [PS2].) Because of the Koszul results proved in [AJS], we conjecture that a Koszul algebra structure also exists for the quasi-hereditary algebras associated to the group-schemes G_1T discussed in [CPS5], whenever $p > h$ and the Lusztig conjecture is valid. (These would be the algebras A_Ω introduced in (5.4), in the special case $r = 1$.)

and satisfies (3.5.1). The following parity condition holds:

$$\begin{aligned}&\mathrm{Ext}^1_B(L(B,\lambda), L(B,\mu)) \neq 0\\ &\quad \Rightarrow \ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2, \ \forall \lambda, \mu \in \Lambda^+.\end{aligned} \tag{3.9.1}$$

Conversely, assume that we wish to prove the character formula for $\mathcal{C}$. Suppose A has a strong Borel subalgebra B satisfying (3.9.1) (with no assumption that A is Koszul). By (3.8(1)),

$$\mathrm{Ext}^1_A(\Delta(A,\lambda), L(A,\mu)) \neq 0 \tag{3.9.2}$$

$$\Rightarrow \ell(\lambda) - \ell(\mu) \equiv 1 \pmod 2, \ \forall \lambda, \mu \in \Lambda^+. \tag{3.9.2}$$

By [CPS4, (5.4b)], [CPS7, (5.10)], (3.9.2) implies the character formula holds.

As mentioned, a similar program (connecting Kazhdan–Lusztig theory with the existence of what are essentially strong Borel subalgebras with an exact functor $\Psi_!$) had been proposed in [K2, p. 604], whose status is unclear to us. The above remarks provide an independent substitute theory (avoiding any assumptions on the existence of exact Borel subalgebras).

4. Homological, Excellent and Exact Borel Categories

We keep the notation of Section 2. Thus, let $\mathcal{C}$ be a fixed highest weight category with finite weight poset $(\Lambda^+, \leqslant^+)$. We have defined in Section 2, the idea of a Borel category $\mathcal{B} \to \mathcal{C}$. In this section, we present various strengthenings of this notion. As we will see in later sections, these categories do arise in the representation theory of algebraic and quantum groups.

It will often be convenient to work in the language of bounded derived categories. Thus, given an additive, right exact functor $\Psi\colon \mathcal{A} \to \mathcal{E}$ (of highest weight categories), there is a left derived functor $\mathbf{L}\Psi\colon D^b(\mathcal{A}) \to D^b(\mathcal{E})$ of bounded derived categories. If Ψ is an additive, left exact functor, then there is a right derived functor $\mathbf{R}\Psi\colon D^b(\mathcal{A}) \to D^b(\mathcal{E})$. If Ψ is exact, we usually just denote $\mathbf{L}\Psi$ and $\mathbf{R}\Psi$ simply by Ψ. The following result, complementing (2.1), is taken from the [K1, Appendix]. Its proof is an applicaton of (1.3) and (1.4).

(4.1) LEMMA. *Let $\Psi\colon \mathcal{C} \to \mathcal{B}$ be an exact, additive functor having a left adjoint $\Psi_!$. Assume that $\mathcal{B}$ is a highest weight category with weight poset $(\Lambda, \leqslant)$ such that $\Lambda^+ \subseteq \Lambda$. The following statements are equivalent:*

(1) $\Psi\nabla(\mathcal{C},\lambda) \cong \nabla(\mathcal{B},\lambda)$ *for all* $\lambda \in \Lambda^+$;

(2) *For* $\mu \in \Lambda$,

$$\mathbf{L}\Psi_!\Delta(\mathcal{B},\mu) \cong \begin{cases}\Delta(\mathcal{C},\mu), & \mu \in \Lambda^+,\\ 0, & \mu \notin \Lambda^+.\end{cases}$$

Now we present the idea of a homological Borel category for $\mathcal{C}$. It represents a more flexible variation on a similar concept introduced by the second author (at the algebra level, without the 'homological' terminology) in [K1, Appendix]. The word 'homological' is inspired by related terminology in [W1, W2].

(4.2) DEFINITION. Let $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ define a Borel category. We say that $\mathcal{B}$ is a *homological Borel category* if for $\lambda \in \Lambda^+$, $\Psi\nabla(\mathcal{C},\lambda) \cong \nabla(\mathcal{B},\lambda)$.

(4.3) PROPOSITION. *Let $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ define a homological Borel category and suppose that $\leqslant$ directs $\mathcal{B}$. Then $\Psi_!\colon\ \mathcal{B} \to \mathcal{C}$ is exact.*

Proof. The irreducible objects in $\mathcal{B}$ must have the form $L(\mathcal{B},\lambda) = \Delta(\mathcal{B},\lambda)$ for $\lambda \in \Lambda$, and by (4.2), the higher derived functors of $\Psi_!$ vanish on $\Delta(\mathcal{B},\lambda)$. □

In spite of this fact, we do not call such a Borel category exact, leaving the name "exact Borel category" to those Borel categories satisfying stronger conditions (see (4.5)). Note also that there do exist homological Borel categories in which $\Psi_!$ is not exact. That is, we may have a homological Borel category $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ in which $\leqslant$ does not direct $\mathcal{B}$. Although we may change the partial ordering on the weight set Λ of $\mathcal{B}$ so that $\leqslant$ directs $\mathcal{B}$ (see (2.3(3))), after such a change, $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ may no longer define a homological Borel category.

(4.4) *Remark.* Let $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ be an exact, additive functor of highest weight categories $\mathcal{C}$ and $\mathcal{B}$ having weight posets $(\Lambda^+, \leqslant^+)$ and $(\Lambda, \leqslant)$, respectively. Assume that Λ^+ is a subposet of Λ: $\Lambda^+ \subseteq \Lambda$ and that $\leqslant^+ = \leqslant|_{\Lambda^+}$. Assume $\Psi\nabla(\mathcal{C},\lambda) \cong \nabla(\mathcal{B},\lambda)$ and $\Psi\Delta(\mathcal{C},\lambda) \cong \Delta(\mathcal{B},\lambda)$ for all $\lambda \in \Lambda^+$. Then Λ^+ *is an ideal of* Λ *with respect to the minimal partial ordering* $\leqslant_{\min}$*, and* Ψ *induces an equivalence* $\mathcal{C} \cong \mathcal{B}[\Lambda^+]$. To see this, let $\lambda \in \Lambda^+$. There is a unique (up to scalar) nonzero morphism $\varphi\colon \Delta(\mathcal{C},\lambda) \to \nabla(\mathcal{C},\lambda)$, and $\mathrm{Im}(\varphi)$ identifies with $L(\mathcal{C},\lambda)$. A similar statement holds for $\mathcal{B}$. Then $\Psi(\varphi)\colon\ \Psi\Delta(\mathcal{B},\lambda) \to \nabla(\mathcal{B},\lambda)$ has $\Psi L(\mathcal{C},\lambda)$, which is nonzero (see (2.3(1))), as its image. It follows that $\Psi L(\mathcal{C},\lambda) \cong L(\mathcal{B},\lambda)$. Therefore, Ψ takes a $\mathcal{C}$-composition series of $\Delta(\mathcal{C},\lambda)$ (resp., $\nabla(\mathcal{C},\lambda)$) to a $\mathcal{B}$-composition series of $\Delta(\mathcal{B},\lambda)$ (resp., $\nabla(\mathcal{B},\lambda)$). This implies that if $\lambda \in \Lambda^+$ and $\mu \in \Lambda$ with $\mu \leqslant_{\min} \lambda$, then $\mu \in \Lambda^+$. In other words, Λ^+ is an ideal of $(\Lambda, \leqslant_{\min})$. Now, the required result follows from the Comparison Theorem [PS1, Section 5] (see also [CPS2, (1.5)]).

We give several strengthenings of the concept of a homological Borel category. Each definition presents an independent concept, which arises in practice. In particular, part (1) is equivalent to a definition in [K1].

(4.5) DEFINITION. Let $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ define a homological Borel category for $\mathcal{C}$ as in Definition (4.2).

(1) $\mathcal{B}$ is an *exact Borel category* for $\mathcal{C}$ if $(\Lambda, \leqslant) = (\Lambda^+, \leqslant^+)$;
(2) $\mathcal{B}$ is an *excellent Borel category* for $\mathcal{C}$ provided Ψ has a right adjoint Ψ_* and there is a function $(-)^+ \colon \Lambda \to \Lambda^+$, $\lambda \mapsto \lambda^+$, such that, for all $\lambda \in \Lambda$, we have

$$\mathbf{R}\Psi_*\nabla(\mathcal{B}, \lambda) = \nabla(\mathcal{C}, \lambda^+), \tag{4.5.1}$$

and such that the adjunction map

$$\Psi\Psi_*\nabla(\mathcal{B}, \lambda) \to \nabla(\mathcal{B}, \lambda) \quad \text{is surjective.} \tag{4.5.2}$$

(3) $\mathcal{B}$ is a *complete Borel category* provided that the adjunction transformation $\mathbf{L}\Psi_!\Psi \to \mathrm{id}_{D^b(\mathcal{C})}$ is an equivalence of functors $D^b(\mathcal{C}) \to D^b(\mathcal{C})$. (In particular, $\Psi_!\Psi \to \mathrm{id}_{\mathcal{C}}$ is an equivalence of functors $\mathcal{C} \to \mathcal{C}$.)

The next three propositions indicate the main properties of exact, excellent and complete Borel categories, respectively.

(4.6) PROPOSITION. *Let* $\Psi \colon \mathcal{C} \to \mathcal{B}$ *define an exact Borel category for* $\mathcal{C}$. *Then:*

(1) *The partial ordering* $\leqslant$ *directs* $\mathcal{B}$.
(2) *The functor* $\Psi_!$ *is exact.*
(3) *If* $\mathcal{C}'$ *is a direct sum of blocks of* $\mathcal{C}$, *then* $\mathcal{C}'$ *also has an exact Borel category.*
(4) *If* $\Psi \colon \mathcal{C} \to \mathcal{B}$ *defines also a complete Borel category, then* Ψ *is an equivalence of categories.*

Proof. (1) is obvious, since, by (2.2) and (4.4), $\Delta(\mathcal{B}, \lambda)$ is irreducible for all $\lambda \in \Lambda$.

(2) follows from (1) and (4.3).

(3) $\Psi' = \Psi|_{\mathcal{C}'} \colon \mathcal{C}' \to \mathcal{B}$ defines $\mathcal{B}$ as a homological Borel category of $\mathcal{C}'$. Let $\Lambda' \subseteq \Lambda^+ = \Lambda$ be the weight poset of $\mathcal{C}'$. Let $\Gamma = \Lambda \backslash \Lambda'$, and $\mathcal{S}$ be the Serre subcategory of $\mathcal{B}$ generated by the $L(\mathcal{B}, \mu)$ with $\mu \in \Gamma$. Denote by $j^* \colon \mathcal{B} \to \mathcal{B}/\mathcal{S}$ the canonical quotient functor. Then, by (2.5), $j^*\Psi' \colon \mathcal{C}' \to \mathcal{B}/\mathcal{S}$ defines a Borel category, which is also homological, since, for $\lambda \in \Lambda'$, we have

$$j^*\Psi'\nabla(\mathcal{C}', \lambda) \cong j^*\nabla(\mathcal{B}, \lambda) \cong j^*I(\mathcal{B}, \lambda) \cong I(\mathcal{B}/\mathcal{S}, \lambda) \cong \nabla(\mathcal{B}/\mathcal{S}, \lambda).$$

The category $\mathcal{B}/\mathcal{S}$ has weight poset Λ', so it is an exact Borel category of $\mathcal{C}'$.

(4) We have

$$\Psi_!\nabla(\mathcal{B}, \lambda) \cong \Psi_!\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{C}, \lambda)$$

if $\mathcal{B}$ is complete. Also, $\Psi_!\Delta(\mathcal{B}, \lambda) \cong \Delta(\mathcal{C}, \lambda)$ (see (2.1)). The result follows from (4.4), viewing $\Psi_!$ here as Ψ in (4.4). □

(4.7) PROPOSITION. *Let* $\Psi \colon \mathcal{C} \to \mathcal{B}$ *define an excellent Borel category of* $\mathcal{C}$. *Then:*

(1) *For* $\lambda \in \Lambda$, $\lambda \leqslant \lambda^+$, *and the equality holds if and only if* $\lambda \in \Lambda^+$.

(2) *For* $\lambda \in \Lambda^+$, *the adjunction map* $\Psi\Psi_*\nabla(\mathcal{B}, \lambda) \to \nabla(\mathcal{B}, \lambda)$ *is an isomorphism. We have a surjective homomorphism* $\nabla(\mathcal{B}, \mu^+) \to \nabla(\mathcal{B}, \mu)$ *for all* $\mu \in \Lambda$.

(3) *For* $\lambda \in \Lambda^+$, $\Psi\Delta(\mathcal{C}, \lambda)$ *has a multiplicity free* Δ*-filtration in* $\mathcal{B}$. *Any* $\Delta(\mathcal{B}, \mu)$ *with* $\mu \in \Lambda$ *satisfying* $\mu^+ = \lambda$ *occurs precisely once as a section in a* Δ*-filtration of* $\Psi\Delta(\mathcal{C}, \lambda)$. *All* $\mathcal{B}$*-composition factors of* $\Psi\Delta(\mathcal{C}, \lambda)$ *have weights* $\leqslant \lambda$, *and* $L(\mathcal{B}, \lambda)$ *is a multiplicity one composition factor occurring in the socle.*

(4) $\Psi\colon \mathcal{C} \to \mathcal{B}$ *defines a complete Borel category of* $\mathcal{C}$.

(5) *If we use* $\leqslant^+_{\min}$ *and* $\leqslant_{\min}$ *to define the highest weight category structures on* $\mathcal{C}$ *and* $\mathcal{B}$, *respectively, then* $\Psi\colon \mathcal{C} \to \mathcal{B}$ *still defines an excellent Borel category of* $\mathcal{C}$. *Therefore, the above* (1)–(4) *hold when* $\leqslant^+$ *is replaced by* $\leqslant^+_{\min}$ *and* $\leqslant$ *is replaced by* $\leqslant_{\min}$.

Proof. (1) Let $\lambda \in \Lambda$. By (4.5.1) and (4.2),

$$\Psi\Psi_*\nabla(\mathcal{B}, \lambda) \cong \Psi\nabla(\mathcal{C}, \lambda^+) \cong \nabla(\mathcal{B}, \lambda^+). \tag{4.7.1}$$

Then the surjectivity of (4.5.2) implies that $\lambda \leqslant \lambda^+$ (even $\lambda \leqslant_{\min} \lambda^+$). Obviously, if $\lambda \notin \Lambda^+$, $\lambda \neq \lambda^+$. Now let $\lambda \in \Lambda^+$. Since $\mathrm{Hom}_{\mathcal{C}}(\Delta(\mathcal{C}, \lambda), \nabla(\mathcal{C}, \lambda)) \neq 0$,

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{\mathcal{B}}(\Psi\Delta(\mathcal{C}, \lambda), \Psi\nabla(\mathcal{C}, \lambda)) \cong \mathrm{Hom}_{\mathcal{B}}(\Psi\Delta(\mathcal{C}, \lambda), \nabla(\mathcal{B}, \lambda)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\Delta(\mathcal{C}, \lambda), \Psi_*\nabla(\mathcal{B}, \lambda)) \cong \mathrm{Hom}_{\mathcal{C}}(\Delta(\mathcal{C}, \lambda), \nabla(\mathcal{C}, \lambda^+)). \end{aligned}$$

Therefore, by (1.3), $\lambda = \lambda^+$. Thus, (1) has been proved.

(2) follows from (4.7.1).

(3) For $\lambda \in \Lambda^+$, $\mu \in \Lambda$, and integer n,

$$\begin{aligned} &\dim \mathrm{Hom}^n_{D^b(\mathcal{C})}(\Psi\Delta(\mathcal{C}, \lambda), \nabla(\mathcal{B}, \mu)) \\ &\quad = \dim \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{C}, \lambda), \mathbf{R}\Psi_*\nabla(\mathcal{B}, \mu)) \\ &\quad = \dim \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Delta(\mathcal{C}, \lambda), \nabla(\mathcal{C}, \mu^+)) \\ &\quad = \delta_{n,0}\delta_{\lambda,\mu^+}. \end{aligned}$$

From the $n = 1$ case and (1.4), it follows that $\Psi\Delta(\mathcal{C}, \lambda)$ has a Δ-filtration. On the other hand, if $V \in \mathrm{Ob}(\mathcal{B})$ has a Δ-filtration, then $\dim \mathrm{Hom}_{\mathcal{B}}(M, \nabla(\mathcal{B}, \mu))$ calculates the multiplicity of $\Delta(\mathcal{B}, \mu)$ as a section in any Δ-filtration of M. Therefore, any $\Delta(\mathcal{B}, \mu)$ with $\mu \in \Lambda$ satisfying $\mu^+ = \lambda$ occurs precisely once in a Δ-filtration of $\Psi\Delta(\mathcal{C}, \lambda)$. Now by (1) and the fact that all composition factors of $\Delta(\mathcal{B}, \mu)$ have weights $\leqslant \mu$, all $\mathcal{B}$-composition factors of $\Psi\Delta(\mathcal{C}, \lambda)$ have weights $\leqslant \lambda$, and $L(\mathcal{B}, \lambda)$ occurs only once. Also, $L(\mathcal{B}, \lambda)$ must in the socle, otherwise there is a nonsplit extension $0 \to L(\mathcal{B}, \mu) \to E \to L(\mathcal{B}, \lambda) \to 0$ with $\mu < \lambda$. This is impossible, since $L(\mathcal{B}, \lambda) = \Delta(\mathcal{B}, \lambda)$. This completes the proof of (3).

(4) Now we let $\theta\colon \mathbf{L}\Psi_!\Psi \to \mathrm{id}_{D^b(\mathcal{C})}$ be the adjunction transformation. We first note that, under the general setting of a Borel category, $\theta_V\colon \mathbf{L}\Psi_!\Psi V \to V$ is

nonzero for any $0 \neq V \in \mathrm{Ob}(\mathcal{C})$, regarded as a complex concentrated in degree 0. This is because $\Psi V \neq 0$ (see (2.3(1))) and θ_V is the image of $\mathrm{id}_{\Psi V}$ under the isomorphism $\mathrm{Hom}_{\mathcal{B}}(\Psi V, \Psi V) \xrightarrow{\sim} \mathrm{Hom}_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!\Psi V, V)$. To prove (3), we must show that θ_V is an isomorphism for any $V \in \mathrm{Ob}(D^b(\mathcal{C}))$. Because $\mathbf{L}\Psi_!\Psi$ is an exact functor on triangulated categories, a standard truncation argument shows that it suffices to prove that $\theta_{L(\mathcal{C},\lambda)}$ is an isomorphism for all $\lambda \in \Lambda^+$. By (2) and (4.1), $\mathbf{L}\Psi_!\Psi\Delta(\mathcal{C},\lambda) \cong \Delta(\mathcal{C},\lambda)$, so that the nonzero homomorphism $\theta_{\Delta(\mathcal{C},\lambda)}$ is an isomorphism. In particular, $\theta_{L(\mathcal{C},\lambda)}$ is an isomorphism if $\lambda \in \Lambda^+$ is minimal. Now consider a nonminimal $\lambda \in \Lambda^+$, and form the exact sequence $0 \to Q(\lambda) \to \Delta(\mathcal{C},\lambda) \to L(\mathcal{C},\lambda) \to 0$ in $\mathcal{C}$. By induction, $\theta_{Q(\lambda)}$ is an isomorphism, while $\theta_{\Delta(\mathcal{C},\lambda)}$ is an isomorphism. It follows that $\theta_{L(\mathcal{C},\lambda)}$ is an isomorphism. This proves (4).

(5) Replacement of $\leqslant^+$ with $\leqslant^+_{\min}$ and of $\leqslant$ with $\leqslant_{\min}$ do not change ∇- and Δ-objects in both categories $\mathcal{C}$ and $\mathcal{B}$, so it suffices to verify the compatibility of $\leqslant^+_{\min}$ with $\leqslant_{\min}$, i.e., for $\lambda, \mu \in \Lambda^+$, $\mu \leqslant^+_{\min} \lambda$ implies $\mu \leqslant_{\min} \lambda$. We assume $L(\mathcal{C},\mu)$ is a composition factor of $\nabla(\mathcal{C},\lambda)$ or $\Delta(\mathcal{C},\lambda)$. If $L(\mathcal{C},\mu)$ is a composition factor of $\nabla(\mathcal{C},\lambda)$, then $L(\mathcal{B},\mu) \subseteq \Psi L(\mathcal{C},\mu)$ (see $(*)$ in the proof of (2.1)), and $\Psi L(\mathcal{C},\mu)$ is a subquotient of $\Psi\nabla(\mathcal{C},\lambda) \cong \nabla(\mathcal{B},\lambda)$. Thus, $L(\mathcal{B},\mu)$ is a composition factor of $\nabla(\mathcal{B},\lambda)$. Hence, $\mu \leqslant_{\min} \lambda$. If $L(\mathcal{C},\mu)$ is a composition factor of $\Delta(\mathcal{C},\lambda)$, by (3), together with $(*)$, $L(\mathcal{B},\mu)$ is a composition factor of some $\Delta(\mathcal{B},\nu)$ with $\nu^+ = \lambda$. So $\mu \leqslant_{\min} \nu$. As noted in the proof of (1), $\nu \leqslant_{\min} \nu^+ = \lambda$. Therefore, $\mu \leqslant_{\min} \lambda$. □

The following result shows that when $\mathcal{C}$ has a complete Borel category $\mathcal{B}$, $\mathcal{C}$ is fully embedded in $\mathcal{B}$. Thus, the situation is analogous to the case for a reductive algebraic group G, where the category of rational G-modules can be viewed as fully embedded into the category of rational B-modules for a Borel subgroup B. (This depends on the completeness of the variety G/B, which explains our choice of terminology.) When $\mathcal{B}$ is an excellent Borel category, the third part of the result below gives a homological characterization of the strict image of $\mathcal{C}$ in $\mathcal{B}$. As we will see in Section 5, in the case of algebraic groups, excellent Borel categories exist, so that this result gives a new characterization of the category of rational G-modules as a subcategory of the category of rational B-modules. By Section 6, this result extends to the case of 'generalized q-Schur algebras' associated to quantum groups at a root of unity.

(4.8) THEOREM. *Suppose $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines a complete Borel category of $\mathcal{C}$. Then:*

(1) *Ψ is a full embedding of $\mathcal{C}$ (resp., $D^b(\mathcal{C})$) into $\mathcal{B}$ (resp., $D^b(\mathcal{B})$). In particular, for $V, W \in \mathrm{Ob}(\mathcal{C})$, there is a natural isomorphism*

$$\mathrm{Ext}^n_{\mathcal{C}}(V, W) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathcal{B}}(\Psi V, \Psi W)$$

valid for all integers $n \geqslant 0$.

(2) *The triangulated category $D^b(\mathcal{C})$ is equivalent to the quotient of $D^b(\mathcal{B})$ by the épaisse subcategory consisting of all objects $N \in \mathrm{Ob}(D^b(\mathcal{B}))$ satisfying*

$$\mathrm{Hom}^\bullet_{D^b(\mathcal{B})}(N, \nabla(\mathcal{B}, \lambda)) = 0, \quad \forall \lambda \in \Lambda^+. \tag{4.8.1}$$

(3) *Assume that $\mathcal{B}$ is an excellent Borel category. The $M \in \mathrm{Ob}(D^b(\mathcal{B}))$ which are isomorphic to an object of $\Psi(D^b(\mathcal{C}))$ are precisely the objects M satisfying*

$$\mathrm{Hom}^\bullet_{D^b(\mathcal{B})}(\Delta(\mathcal{B}, \nu), M) = 0, \quad \forall \nu \in \Lambda \backslash \Lambda^+. \tag{4.8.2}$$

In particular, $M \in \mathrm{Ob}(\mathcal{B})$ lies (up to isomorphism) in the Ψ-image of $\mathcal{C}$ if and only if

$$\mathrm{Ext}^\bullet_{\mathcal{B}}(\Delta(\mathcal{B}, \nu), M) = 0, \quad \forall \nu \in \Lambda \backslash \Lambda^+. \tag{4.8.3}$$

Proof. Both (1) and (3) need only to be proved for the derived category cases. The first part of (1) follows since $\mathbf{L}\Psi_!\Psi \cong \mathrm{id}_{D^b(\mathcal{C})}$. The embedding is full, since, for $V, W \in \mathrm{Ob}(D^b(\mathcal{C}))$,

$$\mathrm{Hom}_{D^b(\mathcal{B})}(\Psi V, \Psi W) \cong \mathrm{Hom}_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!\Psi V, W) \cong \mathrm{Hom}_{D^b(\mathcal{C})}(V, W).$$

Now the second assertion in (1) follows from the isomorphisms (valid for all $n \geqslant 0$ and all $V, W \in \mathrm{Ob}(\mathcal{C})$):

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{C}}(V, W) &\cong \mathrm{Hom}_{D^b(\mathcal{C})}(V, W[n]) \cong \mathrm{Hom}_{D^b(\mathcal{B})}(\Psi V, \Psi W[n]) \\ &\cong \mathrm{Ext}^n_{\mathcal{B}}(\Psi V, \Psi W). \end{aligned}$$

Next we prove (3). For any $M = \Psi V$, by (4.1),

$$\begin{aligned} &\mathrm{Hom}^\bullet_{D^b(\mathcal{B})}(\Delta(\mathcal{B}, \nu), \Psi V) \\ &\quad \cong \mathrm{Hom}^\bullet_{D^b(\mathcal{C})}(\mathbf{L}\Psi_!\Delta(\mathcal{B}, \nu), V) = 0, \quad \forall \nu \in \Lambda \backslash \Lambda^+. \end{aligned}$$

Conversely, if M satisfies (4.8.2), the adjunction map $\Psi\mathbf{R}\Psi_* M \to M$ is an isomorphism. It suffices to check the induced map

$$\mathrm{Hom}_{D^b(\mathcal{B})}(X, \Psi\mathbf{R}\Psi_* M) \to \mathrm{Hom}_{D^b(\mathcal{B})}(X, M) \tag{4.8.4}$$

is an isomorphism for all $X \in \mathrm{Ob}(D^b(\mathcal{B}))$. Because $\mathcal{B}$ is excellent, $D^b(\mathcal{B})$ is generated by the $\Psi\Delta(\mathcal{C}, \lambda)$, $\lambda \in \Lambda^+$, and the $\Delta(\mathcal{B}, \nu)$, $\nu \in \Lambda \backslash \Lambda^+$. Hence, (4.8.2) implies that to prove (4.8.4) is an isomorphism for all X, it suffices to check, for all integers n and all $\lambda \in \Lambda^+$, that the map

$$\mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C}, \lambda), \Psi\mathbf{R}\Psi_* M) \to \mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C}, \lambda), M) \tag{4.8.5}$$

is an isomorphism. Since

$$\mathrm{Hom}^n_{D^b(\mathcal{B})}(\Psi\Delta(\mathcal{C}, \lambda), M) \cong \mathrm{Hom}^n_{D^b(\mathcal{C})}(\Delta(\mathcal{C}, \lambda), \mathbf{R}\Psi_* M),$$

(4.8.5) has a right inverse defined by Ψ, and, taking the full embedding into account, it is an isomorphism.

Finally, to prove (2), let $\mathcal{E}$ be the full subcategory of $D^b(\mathcal{B})$ consisting of all $N \in \mathrm{Ob}(D^b(\mathcal{B}))$ satisfying (4.8.1). Then $\mathcal{E}$ is épaisse [V], while $\mathbf{L}\Psi_!(\mathcal{E}) = 0$, by properties of the adjoint pair $(\mathbf{L}\Psi_!, \Psi)$ since any nonzero object in $D^b(\mathcal{C})$ has a nonzero morphism to some $\nabla(\mathcal{C}, \lambda)[n]$. Thus, there is a commutative diagram

$$\begin{array}{ccc} D^b(\mathcal{B}) & \xrightarrow{\quad\pi\quad} & D^b(\mathcal{B})/\mathcal{E} \\ & {\scriptstyle \mathbf{L}\Psi_!}\searrow \quad \swarrow{\scriptstyle \tau} & \\ & D^b(\mathcal{C}) & \end{array}$$

in which π is the quotient functor. Now $\pi\Psi$ provides a right inverse to τ, since $\mathcal{B}$ is complete. To show that it also gives a left inverse, we first claim that the adjunction morphism

$$X \to \Psi\mathbf{L}\Psi_! X \tag{4.8.6}$$

has mapping cone C lying in $\mathcal{E}$ for $X \in \mathrm{Ob}(D^b(\mathcal{B}))$. If $\mathbf{L}\Psi_!$ is applied to the map in (4.8.6), it becomes an isomorphism. Thus, $\mathbf{L}\Psi_! C = 0$, so

$$\begin{aligned} \mathrm{Hom}^n_{D^b(\mathcal{B})}(C, \nabla(\mathcal{B}, \lambda)) &\cong \mathrm{Hom}^n_{D^b(\mathcal{B})}(C, \Psi\nabla(\mathcal{C}, \lambda)) \\ &\cong \mathrm{Hom}^n_{D^b(\mathcal{C})}(\mathbf{L}\Psi_! C, \nabla(\mathcal{C}, \lambda)) = 0 \end{aligned}$$

for $\lambda \in \Lambda^+$, $n \in \mathbb{Z}$, as required. Since $\mathbf{L}\Psi_! = \tau\pi$, applying π to (4.8.6) yields an isomorphism $\mathrm{id}_{D^b(\mathcal{C})/\mathcal{E}} \xrightarrow{\sim} \pi\Psi\tau$, proving of (2). □

The following proposition is a strengthening of (2.4).

(4.9) PROPOSITION. *Let* $\Psi\colon \mathcal{C} \to \mathcal{B}$ *define the highest weight category* $\mathcal{B}$, *with weight poset* $(\Lambda, \leqslant)$, *as a homological* (*resp., exact, excellent, complete*) *Borel category of highest weight category* $\mathcal{C}$, *with weight poset* $(\Lambda^+, \leqslant^+)$. *Let* Γ^+ *be an ideal in* $(\Lambda^+, \leqslant^+)$, *and* Γ *an ideal in* $(\Lambda, \leqslant)$ *satisfying* $\Gamma \cap \Lambda^+ = \Gamma^+$ (*and* $\mu^+ \in \Gamma^+$ *for all* $\mu \in \Gamma$, *if* Ψ *defines an excellent Borel category*). *Then* $\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ *and* $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$ *also define homological* (*resp., exact, excellent, complete*) *Borel subcategories.*

Proof. (2.4) gives the required functors between the categories under consideration, except when Ψ defines an excellent Borel category. If $\mathcal{B}$ is an excellent Borel category, we must to show Ψ_* carries objects in $\mathcal{C}[\Gamma^+]$ to objects in $\mathcal{B}[\Gamma]$: Since $(-)^+\colon \Gamma \to \Gamma^+$, by (4.5.1), $\Psi_*\nabla(\mathcal{B}, \mu) \in \mathrm{Ob}(\mathcal{C}[\Gamma^+])$. Also, Ψ_* is left exact, so $L(\mathcal{B}, \mu)$, as a subobject of $\nabla(\mathcal{B}, \mu)$, is sent by Ψ_* into $\mathcal{C}[\Gamma^+]$. This gives $\Psi_*|_{\mathcal{B}[\Gamma]}\colon \mathcal{B}[\Gamma] \to \mathcal{C}[\Gamma^+]$, which is right adjoint to $\Psi|_{\mathcal{C}[\Gamma^+]}$. By (1.1), Ψ_* induces a functor $\overline{\Psi}_*\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$, serving as a right adjoint to $\overline{\Psi}$.

Now $\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ inherits the defining conditions given in (4.2), (4.5) from the corresponding conditions for $\Psi\colon \mathcal{C} \to \mathcal{B}$. Therefore, if $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines a homological (resp., exact, excellent, complete) Borel category, so does

$\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$. If $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines a homological or exact Borel category, then $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$ also defines a homological or exact Borel category.

Now let $J^*\colon \mathcal{C} \to \mathcal{C}(\Lambda^+\backslash\Gamma^+)$ (resp., $j^*\colon \mathcal{B} \to \mathcal{B}(\Lambda\backslash\Gamma)$) be the canonical quotient functor, and consider $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$, where $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines a complete Borel category. We claim that

$$J^*(\mathbf{L}\Psi_!) \cong (\mathbf{L}\overline{\Psi}_!)j^*. \tag{4.9.1}$$

Since J^* and j^* are exact, and $J^*\Psi_! = \overline{\Psi}_! j^*$, we have natural transformations $(\mathbf{L}\overline{\Psi}_!)j^* \to \mathbf{L}(\overline{\Psi}_! j^*) = \mathbf{L}(J^*\Psi_!) \xrightarrow{\sim} J^*(\mathbf{L}\Psi_!)$. Denote by $\theta\colon (\mathbf{L}\overline{\Psi}_!)j^* \to J^*(\mathbf{L}\Psi_!)$ the composition of these natural transformations. First, we show that $\theta_{\Delta(\mathcal{B},\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$. If $\lambda \in \Gamma$, then, by (4.1), $J^*(\mathbf{L}\Psi_!)\Delta(\mathcal{B},\lambda) = 0 = (\mathbf{L}\overline{\Psi}_!)j^*\Delta(\mathcal{B},\lambda)$. If $\lambda \in \Lambda\backslash\Gamma$, then each term in the minimal projective resolution of $\Delta(\mathcal{B},\lambda)$ is a direct sum of objects of the form $P(\mathcal{B},\mu)$ with $\mu \in \Lambda\backslash\Gamma$. Since for such a μ, $j^*P(\mathcal{B},\mu) \cong P(\mathcal{B}(\Lambda\backslash\Gamma),\mu)$, so j^* carries the minimal projective resolution of $\Delta(\mathcal{B},\lambda)$ to a projective resolution of $\Delta(\mathcal{B}(\Lambda\backslash\Gamma),\lambda)$. This means that $\theta_{\Delta(\mathcal{B},\lambda)}$ is an isomorphism. So we obtain that $\theta_{\Delta(\mathcal{B},\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$, as required. In particular, if $\lambda \in \Lambda$ is minimal, $\theta_{L(\mathcal{B},\lambda)}$ is an isomorphism. If $\lambda \in \Lambda$ is not minimal, we have an exact sequence $0 \to Q(\mathcal{B},\lambda) \to \Delta(\mathcal{B},\lambda) \to L(\mathcal{B},\lambda) \to 0$. Since all involved functors are exact on $D^b(\mathcal{B})$, we assume that $\theta_{Q(\mathcal{B},\lambda)}$ is an isomorphism. It follows that $\theta_{L(\mathcal{B},\lambda)}$ is an isomorphism for all $\lambda \in \Lambda$. This implies (4.9.1), by the exactness of involved functors.

If $\Psi\colon \mathcal{C} \to \mathcal{B}$ defines an excellent Borel category, by a similar argument, using $\nabla(\mathcal{B},\lambda)$ (and injective resolutions),

$$J^*(\mathbf{R}\Psi_*) \cong (\mathbf{R}\overline{\Psi}_*)j^*. \tag{4.9.2}$$

If $\mathcal{B}$ is a complete Borel category, by (4.9.1),

$$\mathbf{L}\overline{\Psi}_!\overline{\Psi} \cong \mathbf{L}\overline{\Psi}_!\overline{\Psi}J^*J_* = (\mathbf{L}\overline{\Psi}_!)j^*\Psi J_* \cong J^*\mathbf{L}\Psi_!\Psi J_* \cong J^*J_* \cong \mathrm{id}_{D^b(\mathcal{C}(\Lambda^+\backslash\Gamma^+))},$$

so $\overline{\Psi}$ defines $\mathcal{B}(\Lambda\backslash\Gamma)$ as a complete Borel category of $\mathcal{C}(\Lambda^+\backslash\Gamma^+)$.

If $\mathcal{B}$ is an excellent Borel category of $\mathcal{C}$, by (4.9.2), for $\lambda \in \Lambda\backslash\Gamma$,

$$\begin{aligned}\mathbf{R}\overline{\Psi}_*\nabla(\mathcal{B}(\Lambda\backslash\Gamma),\lambda) &\cong (\mathbf{R}\overline{\Psi}_*)j^*\nabla(\mathcal{B},\lambda) \cong J^*\mathbf{R}\Psi_*\nabla(\mathcal{B},\lambda)\\ &\cong J^*\nabla(\mathcal{C},\lambda^+) \cong \nabla(\mathcal{C}(\Lambda^+\backslash\Gamma^+),\lambda^+).\end{aligned}$$

The surjectivity of the adjunction map is inherited by the quotient categories. Thus, $\overline{\Psi}\colon \mathcal{C}(\Lambda^+\backslash\Gamma^+) \to \mathcal{B}(\Lambda\backslash\Gamma)$ defines an excellent Borel category. □

(4.10) *Remark.* In (4.9), we make the assumption, for the case $\Phi\colon \mathcal{C} \to \mathcal{B}$ defining an excellent Borel category, that $\mu^+ \in \Gamma^+$, for all $\mu \in \Gamma$. This condition holds if Γ is the ideal of Λ generated by Γ^+, and if the map $(-)^+\colon \Lambda \to \Lambda^+$ is ordering-preserving. If $(-)^+$ is ordering-preserving, and if Γ is generated by Γ^+, then for $\mu \in \Gamma$, $\lambda \in \Gamma^+$ such that $\mu \leqslant \lambda$. Thus, $\mu^+ \leqslant^+ \lambda^+ = \lambda$, hence $\mu^+ \in \Gamma^+$. The argument also says $\Gamma \cap \Lambda^+ = \Gamma^+$.

The condition that $(-)^+$ is ordering-preserving is satisfied in many natural examples arising from representation theory of algebraic groups and quantum groups, provided a suitable partial ordering is used. See (5.6) and Section 6. In general, suppose $\leqslant = \leqslant_{\min}$ and $\leqslant^+ = \leqslant|_{\Lambda^+}$ (which is a refinement of $\leqslant^+_{\min}$, see (4.7(5)) and its proof). If

$$[\Psi V : L(\mathcal{B}, \nu)] = [\Psi V : L(\mathcal{B}, \nu^+)], \quad \forall V \in \mathrm{Ob}(\mathcal{C}),\ \nu \in \Lambda, \tag{4.10.1}$$

then $(-)^+$ must be ordering-preserving. To see this, suppose $\lambda, \mu \in \Lambda$ with $\mu \leqslant \lambda$. We assume $L(\mathcal{B}, \mu)$ is a composition factor of $\nabla(\mathcal{B}, \lambda)$ or $\Delta(\mathcal{B}, \lambda)$. If $L(\mathcal{B}, \mu)$ is a composition factor of $\nabla(\mathcal{B}, \lambda)$, it must be a composition factor of $\nabla(\mathcal{B}, \lambda^+) \cong \Psi\nabla(\mathcal{C}, \lambda^+)$, by (4.7(2)). Now (4.10.1) insures $L(\mathcal{B}, \mu^+)$ is a composition factor of $\Psi\nabla(\mathcal{C}, \lambda^+)$. Hence, $\mu^+ \leqslant \lambda^+$, i.e., $\mu^+ \leqslant^+ \lambda^+$. If $L(\mathcal{B}, \mu)$ is a composition factor of $\Delta(\mathcal{B}, \lambda)$, by (4.7(3)), it must be a composition factor of $\Psi\Delta(\mathcal{C}, \lambda^+)$. Then $L(\mathcal{B}, \mu^+)$ is a composition factor of $\Psi\Delta(\mathcal{C}, \lambda^+)$, hence a composition factor of $\Delta(\mathcal{B}, \nu)$ for some $\nu \in \Lambda$ with $\nu^+ = \lambda^+$, by (4.7(3)) again. Hence $\mu^+ \leqslant \nu \leqslant \nu^+ = \lambda^+$, as required. (Note also that the condition (4.10.1) is a generalization of the fact in the representation theory of algebraic groups, quantum groups, etc. that the weight set (with multiplicities) of a module is invariant under the action of the Weyl group.)

5. Exact and Excellent Borel Subalgebras

We now study the issue of realizing $\mathcal{C}$ and $\mathcal{B}$ as module categories A-mod and B-mod for finite-dimensional (quasi-hereditary) algebras A and B, respectively, so that B is a *subalgebra* of A and Ψ is induced by the natural restriction of rings. We will also give some examples arising from algebraic groups and quantum groups.

(5.1) DEFINITION. Let A be a quasi-hereditary algebra, with weight poset $(\Lambda^+, \leqslant^+)$. Let B be quasi-hereditary subalgebra, with weight poset $(\Lambda, \leqslant)$. We call B a *homological* (resp., *exact*, *excellent*, *complete*) *Borel subalgebra* of A, provided $\Psi = |_B$: A-mod $\to$ B-mod defines a homological (resp., exact, excellent, complete) Borel category for A-mod.

The notion of an exact Borel subalgebra B of a quasi-hereditary algebra A has been defined by König [K1]. One requires that B be a directed algebra, $\Lambda^+ = \Lambda$, and that $\Psi_! = A \otimes_B -$ be exact and carry irreducible B-modules $L(B, \lambda)$ to standard modules $\Delta(A, \lambda)$.

By (2.6), given a Borel category $\mathcal{B}$ associated to $\mathcal{C}$, defined by Ψ: $\mathcal{C} \to \mathcal{B}$, there exist quasi-hereditary algebras A and B, together with an algebra homomorphism ι: $B \to A$, so that $\mathcal{C} \cong A$-mod, $\mathcal{B} \cong B$-mod, and Ψ identifies with ι^*. The morphism $\iota\colon B \to A$ need not be an injection (and so need not identify B with a subalgebra of A). There are two important cases in which B does identify with a subalgebra of A.

(5.2) THEOREM. *Assume that $\mathcal{B}$ is a Borel category associated to the highest weight category $\mathcal{C}$. Let*

$$B = \mathrm{End}_{\mathcal{B}}(P)^{\mathrm{op}} \quad \textit{and} \quad A = \mathrm{End}_{\mathcal{C}}(\Psi_! P)^{\mathrm{op}}$$

for some projective generator P of $\mathcal{B}$. Then the map $\iota\colon B \to A$, obtained by putting $\iota(b) = \Psi_!(b)$, is an injective algebra homomorphism provided that $\mathcal{B}$ is either an exact or an excellent Borel category.

Proof. By (2.6), we assume that $\mathcal{B} = B$-mod, $\mathcal{C} = A$-mod, and $\Psi = \iota^*$ for an algebra homomorphism $\iota\colon B \to A$. Suppose first that $\mathcal{B}$ is an exact Borel category for $\mathcal{C}$. Then $(\Lambda^+, \leqslant^+) = (\Lambda, \leqslant)$. For $\lambda \in \Lambda$, $\Psi\nabla(A, \lambda) \cong \nabla(B, \lambda)$. Thus, if $b \in B$ is such that $\iota(b) = 0$, then b annihilates each $\nabla(B, \lambda)$. But B-mod is directed by $\leqslant$ (see (4.6(1))), so that, given $\lambda \in \Lambda$, $\nabla(B, \lambda) \cong I(B, \lambda)$. So, b annihilates each injective module $I(B, \lambda)$. Hence, b annihilates an injective generator for B-mod, so that $b = 0$, as required.

Now assume that $\mathcal{B}$ is an excellent Borel category for $\mathcal{C}$, and again let $b \in B$ lie in the kernel of ι. As in the exact case above, it is enough to show that b annihilates each injective module $I(B, \lambda)$, $\lambda \in \Lambda$. Because Ψ_* has an exact left adjoint Ψ, $\Psi_* I(B, \lambda)$ is an injective A-module. Since b annihilates $\Psi\Psi_* I(B, \lambda)$ by assumption, it suffices to show that $I(B, \lambda)$ is a homomorphic image of $\Psi\Psi_* I(B, \lambda)$. But for any $\mu \in \Lambda$, the adjunction map $\Psi\Psi_*\nabla(B, \mu) \to \nabla(B, \mu)$ is surjective, by (4.5.2), while the direct images $\mathbf{R}^n\Psi_*\nabla(B, \mu)$ vanish for $n > 0$. Since $I(B, \lambda)$ has a B-module filtration with sections of the form $\nabla(B, \mu)$, an induction argument establishes that $\Psi\Psi_* I(B, \lambda) \to I(B, \lambda)$ is surjective. Hence, $b = 0$. □

The above theorem says that if $\mathcal{B}$ is an exact (resp., excellent) Borel category for $\mathcal{C}$, then $\mathcal{C} \cong A$-mod for some quasi-hereditary algebra A having an exact (resp., excellent) Borel subalgebra B with $\mathcal{B} \cong B$-mod.

(5.3) PROPOSITION. *Let B (with weight poset $(\Lambda, \leqslant)$) be a Borel subalgebra of a quasi-hereditary algebra A (with weight poset $(\Lambda^+, \leqslant^+)$). Write $\mathcal{C} = A$-mod and $\mathcal{B} = B$-mod. Let Γ^+ be an ideal in Λ^+, and Γ be the ideal in Λ generated by Γ^+. Let J be the annihilator in A of all A-modules having only composition factors of the form $L(A, \gamma)$ for $\gamma \in \Gamma^+$. Whenever an excellent Borel subalgebra is in consideration, assume that the map $(-)^+\colon \Lambda \to \Lambda^+$ is ordering-preserving. Then:*

(1) *If B is either exact or excellent, then $\mathcal{C}[\Gamma^+] \cong A/J$-mod and $\mathcal{B}[\Gamma] \cong B/(B \cap J)$-mod.*
(2) *If B is exact, then $B/(B \cap J)$ is an exact Borel subalgebra of A/J.*

(3) *If B is excellent, then $B/(B \cap J)$ is an excellent Borel subalgebra of A/J.*

Proof. By (4.9), if $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ defines an exact (resp., excellent) Borel category, then $\Psi|_{\mathcal{C}[\Gamma^+]}\colon\ \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ defines $\mathcal{B}[\Gamma]$ as an exact (resp., excellent) Borel category. Let J' be the annihilator in B of all objects in $\mathcal{B}[\Gamma]$. From the standard results for quasi-hereditary algebras we obtain $\mathcal{C}[\Gamma^+] \cong A/J$-mod and $\mathcal{B}[\Gamma] \cong B/J'$-mod. To complete the proof, it suffices to show $B \cap J = J'$. The fact Ψ sends $\mathcal{C}[\Gamma^+]$ to $\mathcal{B}[\Gamma]$ implies $J' \subseteq B \cap J$. If B is an exact Borel subalgebra, the inverse inclusion is clear: $B \cap J$ annihilates $\Psi\nabla(A, \lambda) \cong \nabla(\mathcal{B}, \lambda) \cong I(\mathcal{B}, \lambda)$ for all $\lambda \in \Gamma = \Gamma^+$, so $B \cap J \subseteq J'$. If B is an excellent Borel subalgebra, recall the argument in the proof of (5.2): For any $\lambda \in \Gamma$, the B-homomorphism $\Psi\Psi_* I(\mathcal{B}[\Gamma], \lambda) \to I(\mathcal{B}[\Gamma], \lambda)$ is surjective. Since $B \cap J$ annihilates $\Psi\Psi_* I(\mathcal{B}[\Gamma], \lambda)$, it annihilates $I(\mathcal{B}[\Gamma], \lambda)$. So $B \cap J \subseteq J'$. Hence, $B \cap J = J'$ in both cases. This completes the proof. $\square$

(5.4) EXAMPLE. We assume that $\mathsf{K} = \overline{\mathbb{F}}_p$, the algebraic closure of the prime field of characteristic $p > 0$. Let Φ be a root system with weight lattice X. Let G be a semisimple, simply connected algebraic group with root system Φ, defined and split over $\mathbb{F}_p$. Let $F\colon\ G \to G$ be the Frobenius morphism. Fix a maximal split torus T. Then X identifies with the character group $X(T)$ of T. Let B be a Borel subgroup containing T, corresponding to a fixed set of *positive* roots.

For an integer $r > 0$, we form the group schemes $G_rT = (F^r)^{-1}(T)$ (the pull-back of T through the rth power of the Frobenius morphism) and $B_rT = (F^r|_B)^{-1}(T)$. Let $\mathcal{C}_r$ be the category of finite-dimensional rational G_rT-modules. Similarly, let $\mathcal{B}_r$ be the category of finite-dimensional rational B_rT-modules. Though they have infinite weight poset, we have:

(1) Both $\mathcal{C}_r$ and $\mathcal{B}_r$ are highest weight categories with the same poset X, given its ordinary partial ordering: $\mu \leqslant \lambda \Leftrightarrow \lambda - \mu$ is a sum of positive roots.
(2) The category $\mathcal{B}_r$ is directed by this partial ordering. (See [CPS2].)
(3) If $\Psi\colon\ \mathcal{C}_r \to \mathcal{B}_r$ is the restriction functor, then $\Psi\nabla(\mathcal{C}_r, \lambda)$ has B_rT-socle $L(\mathcal{B}_r, \lambda)$, hence it is a nonzero subobject of $I(\mathcal{B}_r, \lambda) = \nabla(\mathcal{B}_r, \lambda)$.
(4) Ψ has an exact left adjoint $\Psi_!$ which satisfies the condition $j_! L(\mathcal{B}_r, \lambda) \cong \Delta(\mathcal{C}_r, \lambda)$ for all $\lambda \in X$.

Thus, $\mathcal{B}_r$ satisfies all the axioms for an exact Borel category, except for the finiteness condition on the weight poset and that of $\mathcal{C}_r$.*

In order to have an exact Borel category, we use the standard constructions for obtaining highest weight category with finite posets from those with infinite poset.

* One can define Borel categories, also homological, exact, excellent and complete Borel categories, without the requirement that Λ is finite. We can avoid the assumption of the existence of a left adjoint functor to Ψ, when not in the complete Borel category case. (See (4.6) below.) A theory can also be developed with this definition. Thus, we say that $\Psi\colon\ \mathcal{C}_r \to \mathcal{B}_r$ defines $\mathcal{B}_r$ as an exact Borel category of $\mathcal{C}_r$.

That is, first form the full subcategory with respect to a finite generated ideal, and then form the quotient category with respect to a finite coideal.

Let $\Gamma \subset X$ be a finitely generated ideal, and let $\mathcal{C}_r^\Gamma = \mathcal{C}_r[\Gamma]$ and $\mathcal{B}_r^\Gamma = \mathcal{B}_r[\Gamma]$. The functor $\Psi|_{\mathcal{C}_r^\Gamma}\colon \mathcal{C}_r^\Gamma \to \mathcal{B}_r^\Gamma$ has an exact left adjoint $\Psi_!|_{\mathcal{C}_r^\Gamma}$, taking Δ-objects to Δ-objects. Let $\Omega \subset \Gamma$ be a finite coideal. Then $\Psi|_{\mathcal{C}_r^\Gamma}$ (resp., $\Psi_!|_{\mathcal{C}_r^\Gamma}$) induces a functor $\overline{\Psi}\colon \mathcal{C}_r^\Gamma(\Omega) \to \mathcal{B}_r^\Gamma(\Omega)$ (resp., $\overline{\Psi}_!\colon \mathcal{B}_r^\Gamma(\Omega) \to \mathcal{C}_r^\Gamma(\Omega)$). As is seen in (1.1), $\overline{\Psi}_!$ is still a left adjoint functor of $\overline{\Psi}$. The above conditions (1)–(4) hold, replacing Ψ by $\overline{\Psi}$. Since $\mathcal{C}_r^\Gamma$ and $\mathcal{B}_r^\Gamma$ have $(\Omega, \leqslant|_\Omega)$ as their weight poset, we obtain:

The functor $\overline{\Psi}\colon \mathcal{C}_r^\Gamma(\Omega) \to \mathcal{B}_r^\Gamma(\Omega)$ *defines* $\mathcal{B}_r^\Gamma(\Omega)$ *as an exact Borel category of* $\mathcal{C}_r^\Gamma(\Omega)$. *Therefore, there exists a quasi-hereditary algebra* $A = A_\Omega$ *such that* A-mod $\cong \mathcal{C}_r^\Gamma(\Omega)$ *and such that* A *has an exact Borel subalgebra.*

Furthermore, by (4.6(3)), if $\mathcal{D}$ is a direct summand (i.e., a sum of blocks) of $\mathcal{C}_r^\Gamma(\Omega)$, then there exists a quasi-hereditary algebra $A_\mathcal{D}$ such that $\mathcal{D} \cong A_\mathcal{D}$-mod and such that $A_\mathcal{D}$ has an exact Borel subalgebra.

(5.5) EXAMPLE. There are several ways in which (5.4) can be quantized. For example, in type A_{n-1}, consider the quantum general linear group $G_q = \mathrm{GL}_q(n)$ defined over an arbitrary algebraically closed field K. Let T_q be the maximal torus in G_q consisting of "diagonal matrices". Then T_q (which is isomorphic to an ordinary algebraic torus $\mathbb{G}_m^{\times n}$) has character group $X = X(T_q)$ indexed by compositions $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers. A weight $\lambda \in X$ is dominant if and only if λ is partition (i.e., $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$). The set X_+ of dominant weights indexes the irreducible G_q-modules. (See [PW, (8.7)] for more details.)

Now assume that $q \in \mathsf{K}$ is a primitive lth root of unity for an *odd* positive integer l. As proved in [PW, (7.2)], there is a Frobenius morphism $F\colon G_q \to G$, where $G = \mathrm{GL}(n)$ is the reductive group of $n \times n$ invertible matrices over k. Letting $T = F(T_q)$ (the standard maximal torus of G), we can form the pull-back $G_{q,1}{\cdot}T = F^{-1}(T)$ of T through F. Similarly, if B_q is the Borel subgroup of G_q [PW, (6.1)] (of "upper triangular matrices"), we let $B_{q,1}{\cdot}T = (F|_{B_q})^{-1}(T)$. It is proved in [PW, (9.8)] that both the categories $\mathcal{C}_{q,1} = G_{q,1}{\cdot}T$-mod and $\mathcal{B}_{q,1} = B_{q,1}{\cdot}T$-mod are highest weight categories with weight poset X (given its usual poset structure $\leqslant$). The evident analogues of (1)–(4) in (5.4) hold for the restriction functor $\Psi\colon \mathcal{C}_{q,1} \to \mathcal{B}_{q,1}$, so we can conclude:

Let $\Gamma \subset X$ *be a finitely generated ideal, and let* $\Omega \subset \Gamma$ *be a finite coideal in* Γ. *The functor* $\Psi\colon \mathcal{C}_{q,1} \to \mathcal{B}_{q,1}$ *induces a functor* $\overline{\Psi}\colon \mathcal{C}_{q,1}^\Gamma(\Omega) \to \mathcal{B}_{q,1}^\Gamma(\Omega)$ *defining* $\mathcal{B}_{q,1}(\Omega)$ *as an exact Borel category of* $\mathcal{C}_{q,1}$. *Therefore, there exists a quasi-hereditary algebra* $A = A_{q,\Omega}$ *such that* $A_{q,\Omega}$-mod $\cong \mathcal{C}_{q,1}^\Gamma(\Omega)$ *and such that* A *has an exact Borel subalgebra.*

Of course, $\mathcal{C}_{q,1}^\Gamma(\Omega) = \mathcal{C}_{q,1}[\Gamma]/\mathcal{C}_{q,1}[\Gamma\backslash\Omega]$, and $\mathcal{B}_{q,1}^\Gamma(\Omega)$ is defined similarly as a quotient category.

Next, let $\mathfrak{g}$ be a complex semisimple Lie algebra with root system Φ. Then we can form the quantum enveloping algebras $\mathfrak{U}_q = \mathfrak{U}_q(\mathfrak{g})$ (divided power form) over a field K as discussed in [APW1]. Fix an odd prime p, which must be > 3 if Φ has a component of type G_2 and require that $q \in \mathsf{K}$ be a primitive lth root of unity for $l = p^e$, $e > 0$, and that K have characteristic 0. We assume $\mathsf{K} = \mathbb{C}$. (Other fields K of characteristic 0 work well, provided $\mathfrak{g}$ is replaced by $\mathfrak{g}_{\mathsf{K}}$, the Lie algebra over K obtained by base-change via a $\mathbb{Z}$-form of $\mathfrak{g}$.) Let $\mathfrak{B}_q$ and $\mathfrak{H}_q$ be the "Borel"* subalgebra corresponding to positive roots and the maximal toral subalgebra of $\mathfrak{U}_q$, respectively, defined in the usual way. Then the weight lattice X is canonically regarded as a subset of the character group $X(\mathfrak{H}_q)$. There is a Frobenius morphism $F\colon \mathfrak{U}_q \to U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$), and we can consider the subalgebra $\mathbf{u}_q{\cdot}\mathfrak{h} = F^{-1}(\mathfrak{h})$, for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ corresponding to the toral subalgebra $\mathfrak{H}_q$ of $\mathfrak{U}_q$. Note that $\mathfrak{H}_q$ is a subalgebra of $\mathbf{u}_q{\cdot}\mathfrak{h}$. Let $\mathcal{C}_{q,1}$ be the category of $\mathbf{u}_q{\cdot}\mathfrak{h}$-modules which have $\mathfrak{H}_q$-weights in X. Similarly, form the module category $\mathcal{B}_{q,1}$ for $\mathbf{b}_q{\cdot}\mathfrak{h} = (F|_{\mathfrak{B}_q})^{-1}(\mathfrak{h})$ associated to $\mathfrak{B}_q$. Using standard results (proved in [APW2]), we establish that $\mathcal{C}_{q,1}$ and $\mathcal{B}_{q,1}$ are highest weight categories with infinite weight poset X with the usual poset structure. Then the above statement concerning exact Borel subalgebras for G_q remains valid.

Finally, if the restrictions on q and/or the characteristic of K in the previous paragraph are dropped, we conjecture that the conclusion of the previous paragraph remains valid, under some mild arithmetic restrictions.

(5.6) EXAMPLE. Let Φ be a root system with positive roots Φ^+. Let $X_+ \subset X$ the set of dominant weights, and W the Weyl group of Φ. For $\lambda \in X$, we denote by λ^+ the unique weight in $W\lambda \cap X_+$ (thus, we have a map $(-)^+\colon X \to X^+$), and let $\lambda^- = w_0\lambda^+$, where w_0 is the longest element in W. The *excellent partial ordering* on X is defined by the rule that $\mu \leqslant_{\mathrm{e}} \lambda$ if $\mu^+ < \lambda^+$ in the ordinary partial ordering or, if $\mu = z\lambda^-$ and $\lambda = w\lambda^-$ ($z, w \in W$), then $z \leqslant w$ in the Chevalley (Bruhat) ordering. For $\lambda, \mu \in X_+$, $\mu \leqslant_{\mathrm{e}} \lambda \Leftrightarrow \mu \leqslant \lambda$ (in the ordinary partial ordering). As is well-known that $\mu \leqslant \mu^+$ for all $\mu \in X$, we also have $\mu \leqslant_{\mathrm{e}} \lambda \Rightarrow \mu \leqslant \lambda$ for all $\mu \in X$ and $\lambda \in X_+$. Note that any finite generated ideal in X with respect to the excellent ordering is finite. The map $(-)^+$ is ordering-preserving (with respect to the excellent partial orderings).

Let G be a semisimple, simply connected algebraic group over an arbitrary algebraically closed field K, and let B be the Borel subgroup of G corresponding to the positive roots. Let $\mathcal{C}$ and $\mathcal{B}$ be the category of rational G-modules and the category of rational B-modules, respectively. The restriction functor $\Psi\colon \mathcal{C} \to \mathcal{B}$ has a right adjoint functor Ind_B^G. We will write $H^0(-)$ for Ind_B^G, and more generally, $H^i(-) = \mathbf{R}^i\mathrm{Ind}_B^G$. The following result is well-known.

* Here and elsewhere in the sequel, we put "Borel" between quotation marks to make it clear that the involved subalgebra is not a Borel subalgebra. In other words, "Borel" subalgebra is only a commonly received name for the algebra. If (2.7) is extended to allow infinite posets, one can prove that these algebras are Borel subalgebras in a strict sense.

The category $\mathcal{C}$ is a highest weight category with weight poset X_+. It has ∇-objects $\nabla(\mathcal{C}, \lambda) = H^0(\lambda^-)$ and Δ-objects $\Delta(\mathcal{C}, \lambda) = H^0(-\lambda)^*$ (linear dual) for $\lambda \in X_+$.

Given a simple root α, let $P_\alpha \supset B$ be the associated parabolic subgroup P_α. Consider the functor $H^0_\alpha(-) = \mathrm{Ind}_B^{P_\alpha}$ and its derived functors $H^j_\alpha(-)$.

Let $\lambda = w\lambda^- \in X$, and let $w = s_1 s_2 \cdots s_r$ be a reduced expression of w, where $s_i = s_{\alpha_i}$ is the reflection with respect to a simple root α_i. Let $H_i^j(-) = H^j_{\alpha_i}(-)$. It is well-known that the B-module

$$N(\lambda) = H^0_1(H^0_2(\cdots H^0_r(\lambda^-)\cdots)) \tag{5.6.1}$$

is independent of the choice of w and the choice of the reduced expression of w; it depends only on λ. The following important result is quoted from [vdK, (1.6)].

The category $\mathcal{B}$ is a highest weight category with respect to the poset $(X, \leqslant_{\mathrm{e}})$. The ∇-objects are the $\nabla(\mathcal{B}, \lambda) = N(\lambda)$, $\lambda \in X$.

The functor $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ also has the following properties:

(1) If $\lambda \in X_+$, then $\Psi\nabla(\mathcal{C}, \lambda) \cong \nabla(\mathcal{B}, \lambda)$, and $\Delta(\mathcal{B}, \lambda) \cong L(\mathcal{B}, \lambda)$.
(2) The functor $\Psi_* = H^0(-)$ serves as a right adjoint functor to Ψ, and satisfies $\mathbf{R}\Psi_*\nabla(\mathcal{B}, \lambda) \cong \nabla(\mathcal{C}, \lambda^+)$, for all $\lambda \in X$. The adjuntion map $\Psi\Psi_*\nabla(\mathcal{B}, \lambda) \to \nabla(\mathcal{B}, \lambda)$ is surjective.
(3) The category $\mathcal{B}$ is directed by the ordinary partial ordering $\leqslant$ on X.

Therefore, the functor $\Psi\colon\ \mathcal{C} \to \mathcal{B}$ defines an 'excellent Borel category' for $\mathcal{C}$, except that the weight poset is infinite, and that the existence of a left adjoint functor of Ψ is not required.

Now let Γ^+ be a finitely generated (= finite) ideal of X_+, and $\Gamma = W\Gamma^+$ be the ideal of X generated by Γ^+. Clearly, Γ is also finite. The left adjoint functor of $\Psi|_{\mathcal{C}[\Gamma^+]}\colon \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ does exist, since $\mathcal{C}[\Gamma^+]$ and $\mathcal{B}[\Gamma]$ can be realized as categories of modules over finite dimensional algebras, and $\Psi|_{\mathcal{C}[\Gamma^+]}$ is the pull-back along an algebra homomorphism. Therefore, we have the following result.

*The functor $\Psi|_{\mathcal{C}[\Gamma^+]}\colon\ \mathcal{C}[\Gamma^+] \to \mathcal{B}[\Gamma]$ defines $\mathcal{B}[\Gamma]$ as an excellent Borel category of $\mathcal{C}[\Gamma^+]$. Therefore, there exists a quasi-hereditary algebra $A = A_\Gamma$ such that A-*mod $\cong \mathcal{C}[\Gamma^+]$ *and such that A has an excellent Borel subalgebra.*

The next section takes up the question of quantizing this example.

6. Classical and Generalized q-Schur Algebras

Suppose char K is arbitrary, and $q \in K$ is nonzero. The first task in this section is to quantize (5.6), based on [W2]. We work in the set-up of (5.5), but use the excellent ordering on X instead of the ordinary partial ordering. Let $\mathcal{C}_q$ (resp., $\mathcal{B}_q$) be the category of integral modules of type 1 over the algebra $\mathfrak{U}_q$ (resp., $\mathfrak{B}_q$). As is well-known, the restriction functor $\Psi = \mathrm{Res}^{\mathfrak{U}_q}_{\mathfrak{B}_q}\colon \mathcal{C}_q \to \mathcal{B}_q$ has a right adjoint $H^0(-) = \mathrm{Ind}^{\mathfrak{U}_q}_{\mathfrak{B}_q}$, which is the quantum analogue of the functor Ind^G_B in (5.6). We have also the right derived functors $H^j(-) = \mathbf{R}^j\mathrm{Ind}^{\mathfrak{U}_q}_{\mathfrak{B}_q}$.

There is an anti-automorphism $\tau\colon \mathfrak{U}_q \to \mathfrak{U}_q$ defined by interchanging the canonical generators E_i and F_i for all i and fixing the elements in $\mathfrak{H}_q$. Then, for a finite-dimensional $\mathfrak{U}_q$-module V, we can define a $\mathfrak{U}_q$-module structure on its linear dual V^* via τ. That is, $(xf)(v) = f(\tau(x)v)$, for $x \in \mathfrak{U}_q$, $f \in V^*$, $v \in V$. The linear space V^* with this $\mathfrak{U}_q$-module structure will be denoted by V°. For a finite-dimensional module V over a subalgebra $\mathfrak{X}$, we can also define V° as a module for the subalgebra $\tau(\mathfrak{X})$. For example, if V is a finite-dimensional $\mathfrak{B}_q$-module, then V° is a $\mathfrak{B}_q^-$-module, and *vice versa*. Here $\mathfrak{B}_q^-$ is the "Borel" subalgebra corresponding to negative roots.

The following result is well-known (at least under some restrictions on K and q, see [APW1]), or can be obtained from [W2], arguing just as in (6.4).

(6.1) PROPOSITION. *The category $\mathcal{C}_q$ is a highest weight category with weight poset $(X_+, \leqslant|_{X^+} = \leqslant_e|_{X^+})$. It has ∇-objects $\nabla(\mathcal{C}_q, \lambda) = H^0(\lambda^-)$ and Δ-objects $\Delta(\mathcal{C}_q, \lambda) = H^0(\lambda^-)^\circ$ for $\lambda \in X_+$.*

As in (5.6), for any simple root α, one has a minimal parabolic subalgebra $\mathfrak{P}_{q,\alpha}$ containing $\mathfrak{B}_q$. Thus, we have the induction functor $H^0_\alpha(-) = \mathrm{Ind}^{\mathfrak{P}_{q,\alpha}}_{\mathfrak{B}_q}(-)$, and we use (5.6.1) to define, for $\lambda \in X$, a $\mathfrak{B}_q$-module $N(\lambda)$.

However, $N(\lambda)$, $\lambda \in X$, can be defined in a second way: Let $v_\lambda \in \Delta(\mathcal{C}_q, \lambda^+)$ be a (nonzero) weight vector of weight λ. Then $N(\lambda) = (\mathfrak{B}_q^- v_\lambda)^\circ$. In [W2], $N(\lambda)$ is defined in this way, and it can be proved to be equivalent to the above definition by using [W2, (8.3)]. The second definition insures that $N(\lambda)$ is independent of the choice of w and the choice of a reduced expression for w.

(6.2) LEMMA. (1) *If $\lambda \in X_+$, then $N(\lambda) = H^0(\lambda^-)|_{\mathfrak{B}_q}$.*

(2) *If $z \leqslant w$ are elements in W and $\mu = z\lambda^-$, $\lambda = w\lambda^-$, then the evaluation homomorphisms induce a surjective homomorphism $\varphi_{w,z}\colon N(\lambda) \twoheadrightarrow N(\mu)$.*

(3) *The weight space $N(\lambda)_\lambda$ is one-dimensional, and it serves as the $\mathfrak{B}_q$-socle of $N(\lambda)$.*

(4) *All weights μ of $N(\lambda)$ satisfy $\mu \leqslant_e \lambda$; if $\lambda^+ = \mu^+$, then μ is a weight of $N(\lambda)$ if and only if $\mu \leqslant_e \lambda$.*

(5) *$H^0(N(\lambda)) \cong H^0(\lambda^-)$, and the evaluation map $H^0(N(\lambda)) \to N(\lambda)$ is surjective and $H^i(N(\lambda)) = 0$ for $i > 0$.*

Proof. All these results are in [W2] or can be easily derived from [W2]. For property (1), see [W2, (8.7(ii))]. Properties (2), (3) and (4) are immediate from the second definition of $N(\lambda)$ and the fact that $\mathfrak{B}_q^- v_\nu \subseteq \mathfrak{B}_q^- v_\mu$, if $\nu^+ = \mu^+$ and $\nu \leqslant \mu$, see [W2, (3.7)]. Finally, property (5) follows from [W2, (8.7(iii))]. □

A filtration of $V \in \mathrm{Ob}(\mathcal{B}_q)$ with $N(\lambda)$'s as sections is called an *excellent* filtration. To check whether an object in $\mathcal{B}_q$ has an excellent filtration, we need a second class of $\mathfrak{B}_q$-modules – the $\overline{M}(\lambda)$ for $\lambda \in X$. First, define $M(\mu)$ to be the $\mathfrak{B}_q$-submodule of $\Delta(\mathcal{C}_q, \mu^+)$ generated by a (nonzero) weight vector of weight μ. (Since μ and μ^+ is conjugate under the action of the Weyl group, $M(\mu)$ is well-defined.) Also, $M(\nu) \subseteq M(\mu)$ if $\mu \leqslant_{\mathrm{e}} \nu$ (see [W2, (3.7)]). Let $\overline{M}(\mu)$ be the quotient module of $M(\mu)$ modulo its submodule generated by all $M(\nu)$ with $\nu >_{\mathrm{e}} \mu$.

(6.3) LEMMA. *An object V in $\mathcal{B}_q$ has an excellent filtration if and only if*

$$\mathrm{Ext}^1_{\mathcal{B}_q}(\overline{M}(\mu), V) = 0, \quad \forall \mu \in X.$$

Proof. See [W2, (6.12) & (5.12)]. □

Now we have the following result.

(6.4) PROPOSITION. *The category $\mathcal{B}_q$ is a highest weight category with respect to the excellent partial ordering $\leqslant_{\mathrm{e}}$, with $\nabla(\mathcal{B}_q, \lambda) = N(\lambda)$ and $\Delta(\mathcal{B}_q, \lambda) = \overline{M}(\lambda)$.*

Proof. By (6.3), every injective object in $\mathcal{B}_q$ has an excellent filtration. We need also that $\mathrm{Ext}^1_{\mathcal{B}_q}(N(\mu), N(\lambda)) \neq 0 \Rightarrow \lambda <_{\mathrm{e}} \mu$. Let Γ be the ideal of X generated by λ and μ, and $\mathcal{B}_q[\Gamma]$ be the Serre subcategory generated by the $L(\mathcal{B}_q, \nu)$ with $\nu \in \Gamma$. If $\lambda \not<_{\mathrm{e}} \mu$, λ is maximal in Γ, then $N(\lambda)$ is injective in $\mathcal{B}_q[\Gamma]$. (See [W2, (5.2)]: $N(\lambda)$ is the λ-weight space of the natural right action of $\mathfrak{B}_q$ on $A(\Gamma)$, in the notation in [W2], thus a direct summand of $A(\Gamma)$ as a left $\mathfrak{B}_q$-module; while $A(\Gamma)$ is injective in $\mathcal{B}_q[\Gamma]$, since we assume K is a field.) Therefore, $\mathrm{Ext}^1_{\mathcal{B}_q}(N(\mu), N(\lambda)) = 0$, as required. Thus, $\mathcal{B}_q$ is a highest weight category with $\nabla(\mathcal{B}_q, \lambda) = N(\lambda)$.

Finally, by [W2, (6.8)], $\overline{M}(\lambda)$ is universal with respect to the property that it generated by a λ-weight vector and has only weights $\leqslant_{\mathrm{e}} \lambda$. Thus, $\Delta(\mathcal{B}_q, \lambda) = \overline{M}(\lambda)$. □

(6.5) PROPOSITION. *Let Γ^+ be a finite ideal of X_+, and let $\Gamma = W\Gamma^+$ be the ideal of $(X, \leqslant_{\mathrm{e}})$ generated by Γ. Then the functor $\Psi|_{\mathcal{C}_q[\Gamma^+]}\colon \mathcal{C}_q[\Gamma^+] \to \mathcal{B}_q[\Gamma]$ defines $\mathcal{B}_q[\Gamma]$ as an excellent Borel category of $\mathcal{C}_q[\Gamma^+]$. Therefore, there exists a quasi-hereditary algebra $A = A_{\Gamma^+}$ such that A-*mod $\cong \mathcal{C}_q[\Gamma^+]$ *and such that A has an excellent Borel subalgebra.*

We next realize the algebras A_{Γ^+} explicitly as *generalized q-Schur algebras*, working in the cases applicable to both algebraic groups (the case $q = 1$) and of quantum groups. We have no requirement on the characteristic of K.

In [DS1, Sections 3 and 5], Du and Scott first defined the generalized q-Schur algebra $\mathfrak{U}_q\langle\Gamma^+\rangle$ associated to Γ^+. As constructed there, $\mathfrak{U}_q\langle\Gamma^+\rangle$ is a quotient algebra of $\mathfrak{U}_q$ with the property that there is an equivalence $\mathfrak{U}_q\langle\Gamma^+\rangle$-mod $\cong \mathcal{C}_q[\Gamma^+]$

of categories. Similarly, one can form a quotient algebra $\mathfrak{B}_q\langle\Gamma\rangle$ of $\mathfrak{B}_q$ with the property that $\mathfrak{B}_q\langle\Gamma\rangle$-mod $\cong \mathscr{B}_q[\Gamma]$, and one can show there is a natural algebra homomorphism $\mathfrak{B}_q\langle\Gamma\rangle \to \mathfrak{U}_q\langle\Gamma^+\rangle$.

We will give another concrete construction of the $\mathfrak{U}_q\langle\Gamma^+\rangle$ and the "Borel" subalgebras $\mathfrak{B}_q\langle\Gamma\rangle$ – the construction via coordinate algebras.

If C is a coalgebra and V is a right C-comodule, the *coefficient subcoalgebra* $\mathrm{Cf}(V)$ of V is the linear span of the elements in the defining matrix of V. That is, if $\{v_j\}$ is a basis for V, and the structure map sends v_j to $\sum_j v_i \otimes f_{ij}$, then $\mathrm{Cf}(V)$ is the span of these f_{ij}'s. Observe that $\mathrm{Cf}(V)$ is a subcoalgebra of C.

If D is a coalgebra spanned by the set Ξ of its group-like elements, then each $\xi \in \Xi$ spans an irreducible comodule over D, denoted by ξ again. Thus, Ξ indexes the set of isomorphism classes of irreducible D-comodules. Any D-comodule is completely reducible, i.e., it is a direct sum of irreducible D-comodules $\xi \in \Xi$.

(6.6) LEMMA. *Let $\theta\colon C \to D$ be a surjective homomorphism of coalgebras with D spanned by the set Ξ of its group-like elements. Let* comod-C *be the category of right C-comodules. For $\Gamma \subseteq \Xi$, let $\mathcal{O}_\Gamma C \in \mathrm{Ob}(\text{comod-}C)$ be the largest subcomodule of C whose restriction to D (via θ) is a direct sum of irreducible comodules indexed by elements in Γ. Then:*

(1) $\mathrm{Cf}(\mathcal{O}_\Gamma C) = \mathcal{O}_\Gamma C$*, thus, $\mathcal{O}_\Gamma C$ is a subcoalgebra of C.*
(2) *The category* comod-$\mathcal{O}_\Gamma C$ *is the full subcategory of the category* comod-C *consisting of objects whose restrictions to D are direct sums of irreducible comodules indexed by elements in Γ.*

Proof. Denote by Δ and ε the comultiplication and the counit of C. Choose a basis $\{c_j\}$ for $\mathcal{O}_\Gamma C$ such that each c_j spans a D-subcomodule isomorphic to $\gamma_j \in \Gamma$. Let $\Delta(c_j) = \sum_i c_i \otimes f_{ij}$. Then $\theta(f_{ij}) = \delta_{ij}\gamma_j$.

Let V be a C-comodule whose restriction to D is a direct sum of irreducible comodules indexed by elements in Γ. Then V is a subcomodule of a direct sum of copies of $\mathcal{O}_\Gamma C$. Thus, V is a comodule over $\mathrm{Cf}(\mathcal{O}_\Gamma C)$. Conversely, let V be a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule with structure map τ. Let $v \in V$ span a D-subcomodule isomorphic to $\xi \in \Xi$, i.e., $(\mathrm{id} \otimes \theta)\tau(v) = v \otimes \xi$. Since V is a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule, ξ must be a linear combination of γ_j's. It follows that $\xi = \gamma_j \in \Gamma$ for some j, since Ξ is linearly independent in D. Thus, comod-$\mathrm{Cf}(\mathcal{O}_\Gamma C)$ is the full subcategory of comod-C consisting of objects whose restrictions to D are direct sums of irreducible comodules indexed by elements in Γ.

Since $\mathrm{Cf}(\mathcal{O}_\Gamma C)$ is a $\mathrm{Cf}(\mathcal{O}_\Gamma C)$-comodule, the restriction of $\mathrm{Cf}(\mathcal{O}_\Gamma C)$ to D is a direct sum of irreducible comodules indexed by elements in Γ. Thus, $\mathrm{Cf}(\mathcal{O}_\Gamma C) \subseteq \mathcal{O}_\Gamma C$. The opposite inclusion is obvious, since $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$. □

We leave the following straightforward result to the reader.

(6.7) LEMMA. *Let $\theta\colon C \to D$ be a homomorphism of Hopf algebras. Let θ_*:* comod-$D \to$ comod-C *be the induction functor, as defined in* [PW, (2.7)]. *Then*

$\theta_* D \cong C$ (with Δ as the structure map), and θ: $C \to D$ is the evaluation homomorphism.

The coordinate algebra $\mathsf{K}[\mathfrak{U}_q]$ of $\mathfrak{U}_q$ is the union of the $(\mathfrak{U}_q/\mathrm{ann} V)^* \subseteq \mathfrak{U}_q^*$ for all finite-dimensional integral $\mathfrak{U}_q$-modules V of type 1. (See [Lin, Sections 1–2] and [APW1, (1.33)].) It carries the natural structure of a Hopf algebra, and the category of right $\mathsf{K}[\mathfrak{U}_q]$-comodules is isomorphic to the category of left, integral $\mathfrak{U}_q$-modules of type 1. Similarly, we form $\mathsf{K}[\mathfrak{B}_q]$ (see [APW1, (2.4)] for another definition of the algebra $\mathsf{K}[\mathfrak{B}_q]$; two definitions are equivalent by [Lin]). The embedding $\mathfrak{B}_q \to \mathfrak{U}_q$ induces a canonical surjective Hopf algebra homomorphism $\mathsf{K}[\mathfrak{U}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{B}_q]$ (see [APW1, (2.7)]). For the maximal toral subalgebra $\mathfrak{H}_q$, the coordinate algebra $\mathsf{K}[\mathfrak{H}_q]$ is defined as the group algebra of the additive group X. Thus, we have a canonical surjective Hopf algebra homomorphism $\mathsf{K}[\mathfrak{B}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{H}_q]$.

For a finite ideal Γ^+ of X_+, put $\Gamma = W\Gamma^+$. Applying (6.6) to the homomorphisms $\mathsf{K}[\mathfrak{U}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{H}_q]$ and $\mathsf{K}[\mathfrak{B}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{H}_q]$ gives $\mathcal{C}_q[\Gamma^+] \cong$ comod-$\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q]$, and $\mathcal{B}_q[\Gamma^+] \cong$ comod-$\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q]$. In particular, $\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q]$ and $\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q]$ are finite-dimensional. Now $(\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q])^*$ and $(\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q])^*$ are quotient algebras of $\mathfrak{U}_q$ and $\mathfrak{B}_q$, respectively. Therefore,

$$\mathfrak{U}_q\langle\Gamma^+\rangle = \left(\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q]\right)^*, \quad \mathfrak{B}_q\langle\Gamma\rangle = \left(\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q]\right)^*. \tag{6.8}$$

Here we use $\mathcal{O}_\Gamma$ instead of $\mathcal{O}_{\Gamma^+}$ for $\mathsf{K}[\mathfrak{U}_q]$-comodules. This is because a $\mathsf{K}[\mathfrak{U}_q]$-comodule has only composition factors of the form $L(\mathcal{C}_q, \lambda)$ for $\lambda \in \Gamma^+$ if and only if all of its weights are in Γ.

The algebras $\mathfrak{U}_q\langle\Gamma^+\rangle$ and $\mathfrak{B}_q\langle\Gamma\rangle$ are quotient algebras of $\mathfrak{U}_q$ and $\mathfrak{B}_q$, respectively. Also, Ψ: $\mathcal{C}_q \to \mathcal{B}_q$ restricts to a functor $\Psi|_{\mathcal{C}_q[\Gamma^+]}$: $\mathcal{C}_q[\Gamma^+] \to \mathcal{B}_q[\Gamma]$. Hence, the homomorphism $\mathsf{K}[\mathfrak{U}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{B}_q]$ induces a coalgebra homomorphism κ: $\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q] \to \mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q]$ (and so an algebra homomorphism ι: $\mathfrak{B}_q\langle\Gamma\rangle \to \mathfrak{U}_q\langle\Gamma^+\rangle$).

The following lemma, in the special case of algebraic groups, has been essentially obtained in [W1, (3.3)], by a different argument. It says the following in this case: Let G be a semisimple, simply connected algebraic group with Borel subgroup $B = UT$, etc. For $\lambda \in X_+$, consider a function $f \in K[B]$ such that all weights μ in the B-module generated by f satisfy $\mu \leqslant_{\mathrm{e}} \lambda$. Then f has the form $f = g|_B$ for some $g \in K[G]$ such that all composition factors $L(\gamma)$ of the G-module generated by g satisfy $\gamma \leqslant \lambda$. Although the injectivity of the algebra homomorphism ι: $\mathfrak{B}_q\langle\Gamma\rangle \to \mathfrak{U}_q\langle\Gamma^+\rangle$ in (6.9) can be directly proved in a similar way as in (5.2), or obtained from (5.2) by standard techniques of passing from highest weight categories with an infinite poset to those with a finite poset, we still include an alternative proof using coordinate algebras. The two proofs are very similar – they both rely on the fact that any object in $\mathcal{B}[\Gamma]$ with an excellent filtration is a homomorphic image of the restriction of an object in $\mathcal{C}[\Gamma^+]$.

(6.9) LEMMA. *The coalgebra homomorphism κ: $\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{U}_q] \to \mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q]$ is surjective. Thus, the algebra homomorphism ι: $\mathfrak{B}_q\langle\Gamma\rangle \to \mathfrak{U}_q\langle\Gamma^+\rangle$ is injective.*

Proof. By (6.7), $H^0(\mathsf{K}[\mathfrak{B}_q]) = \mathsf{K}[\mathfrak{U}_q]$, and the evaluation map is the canonical surjection $\mathsf{K}[\mathfrak{U}_q] \twoheadrightarrow \mathsf{K}[\mathfrak{B}_q]$. Thus, it suffices to show: (i) $H^0(\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]) \subseteq \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$; and (ii) the evaluation map $H^0(\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]) \to \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]$ is surjective. By (6.4), $\mathcal{B}_q[\Gamma]$ is a highest weight category with poset $(\Gamma, \leqslant_{\mathrm{e}})$ and ∇-objects $N(\lambda)$, so the injective object $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]$ has an excellent filtration. By (6.2(7)), $H^0(\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q])$ has a good filtration with all sections having highest weights in Γ^+. Thus, (i) follows. An induction starting from (6.2(7)) shows that the evaluation map $H^0(V) \to V$ is surjective for any $\mathfrak{B}_q$-module with excellent filtration. This implies (ii). □

Combining (6.5), (6.8) and (6.9) verifies that the $\mathfrak{U}_q\langle\Gamma^+\rangle$ have excellent Borel subalgebras.

(6.10) THEOREM. *With the above notation and assumptions, the algebra $\mathfrak{B}_q\langle\Gamma\rangle$ is an excellent Borel subalgebra of $\mathfrak{U}_q\langle\Gamma^+\rangle$.*

(6.11) EXAMPLE. Now we consider the classical q-Schur algebras and their "Borel" subalgebras. Thus, we assume that Φ has type A_{n-1}. Let $A_q(n)$ be coordinate algebra of the quantum $n \times n$-matrix semigroup with parameter q. That is, $A_q(n)$ is the algebra generated by n^2 generators x_{ij}, $i, j = 1, 2, \dots, n$, subject to a set of well-known (homogeneous) relations (see, e.g., [PW, (3.5)]). In fact, $A_q(n)$ is a bialgebra, graded in the natural way, and the homogeneous component of grade r, denoted $A_q(n, r)$, is a subcoalgebra. Define the classical q-Schur algebra $S_q(n, r)$ to be the dual algebra of $A_q(n, r)$.

Let $B_q(n) = A_q(n)/I_q(n)$ (resp., $B_q^-(n) = A_q(n)/I_q^-(n)$, $T_q(n) = A_q(n)/J_q(n)$), where $I_q(n)$ (resp., $I_q^-(n)$, $J_q(n)$) is the ideal generated by all x_{ij} with $i > j$ (resp., $i < j$, $i \neq j$). These are quotient bialgebras. Also, $T_q(n)$ is spanned by the set of group-like elements $\widetilde{X}$ consisting of monomials $x_{11}^{r_1}x_{22}^{r_2}\cdots x_{nn}^{r_n}$ in the x_{ii}'s. We can also consider the homogeneous components $B_q(n, r)$, $B_q^-(n, r)$ and $T_q(n, r)$ of $B_q(n)$, $B_q^-(n)$, and $T_q(n)$, respectively, and form the dual algebras $S_q^+(n, r) = B_q(n, r)^*$, $S_q^-(n, r) = B_q^-(n, r)^*$, and $S_q^0(n, r) = T_q(n, r)^*$. The set $\widetilde{X}(n, r)$ of group-like elements in $T_q(n, r)$ is stable under the action of the Weyl group $W = \mathfrak{S}_n$, and $x_{11}^{r_1}x_{22}^{r_2}\cdots x_{nn}^{r_n} \in \widetilde{X}(n, r)$ is *dominant* if and only if $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_n$, i.e., $(r_1, r_2, \dots, r_n)$ is a partition of r into at most n parts. The set of dominant elements in $\widetilde{X}(n, r)$ will be denoted by $\widetilde{X}(n, r)_+$.

Let $\mathfrak{U}_q$ be quantized enveloping algebra over K (divided power form) associated with the Lie algebra $\mathfrak{sl}_n$. As before, let $\mathfrak{B}_q$ (resp., $\mathfrak{H}_q$) be the "Borel" (resp., maximal toral) subalgebra $\mathfrak{U}_q$. Then X, the weight lattice of the root system of type A_{n-1}, is canonically embedded into the character group of $\mathfrak{H}$. We have a natural map of $\pi\colon\ \widetilde{X} \to X$, sending $x_{11}^{r_1}x_{22}^{r_2}\cdots x_{nn}^{r_n}$ to $(r_1 - r_2)\omega_1 + (r_2 - r_3)\omega_2 + \cdots + (r_{n-1} - r_n)\omega_{n-1}$, where ω_i are the fundamental dominant weights. Two elements in $\widetilde{X}$ have the same image in X if and only if one of them is obtained from the other by multiplying by a power of $x_{11}x_{22}\cdots x_{nn}$. Thus, π is injective on the set $\widetilde{X}(n, r)$

for any r. Let $\Gamma^+ = \pi(\widetilde{X}(n,r)_+)$, and $\Gamma = \pi(\widetilde{X}(n,r))$. Clearly, Γ^+ is a finite ideal of $(X_+, \leqslant)$, and $\Gamma = W\Gamma^+$.

Now we claim:

$$S_q(n,r) \cong \mathfrak{U}_q\langle\Gamma^+\rangle, \qquad S_q^+(n,r) \cong \mathfrak{B}_q\langle\Gamma\rangle. \tag{6.11.1}$$

We give a proof of (6.11.1) as follows: Let $\widetilde{\mathfrak{U}}_q$ be the quantized enveloping algebra over K (divided power form) of the Lie algebra $\mathfrak{gl}_n$.* Let $\widetilde{\mathfrak{B}}_q$ (resp., $\widetilde{\mathfrak{H}}_q$) be the "Borel" (resp., maximal toral) subalgebra of $\widetilde{\mathfrak{U}}_q$. By [Du, (2.7)], $\widetilde{\mathfrak{U}}_q$ is free over its subalgebra $\mathfrak{U}_q$ with a basis consisting of elements in $\widetilde{\mathfrak{H}}_q$. There is a natural surjective algebra homomorphism $\widetilde{\mathfrak{U}}_q \twoheadrightarrow S_q(n,r)$ for all r, by [Du, (3.4)]. This homomorphism maps $\widetilde{\mathfrak{B}}_q$ onto $S_q^+(n,r)$ (see [Du, (1.5)]). We claim that the restrictions of this homomorphism give surjective algebra homomorphisms

$$\mathfrak{U}_q \twoheadrightarrow S_q(n,r), \qquad \mathfrak{B}_q \twoheadrightarrow S_q^+(n,r). \tag{6.11.2}$$

It suffices to show that the restriction of the above homomorphism to $\mathfrak{H}_q$ gives a surjective homomorphism $\mathfrak{H}_q \to S_q^0(n,r)$. This is clear if we consider the dual coalgebra homomorphism $T_q(n,r) \to \mathsf{K}[\mathfrak{H}_q]$: The coalgebra $T_q(n,r)$ has basis $\widetilde{X}(n,r)$, and the set $\widetilde{X}(n,r)$ is mapped injectively to $X \subseteq \mathsf{K}[\mathfrak{H}_q]$, by the above.

Now the coalgebra homomorphism $A_q(n,r) \to \mathsf{K}[\mathfrak{U}_q]$ dual to (6.11.2) is injective and has image in $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$. Thus, we need only to show that $A_q(n,r)$ and $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$ have the same dimension. Both the categories comod-$A_q(n,r)$ and comod-$\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$ are highest weight categories with weight poset $(\Gamma^+, \leqslant)$ (cf. [PW, (11.5.2)] for comod-$A_q(n,r)$). Under the canonical homomorphism $A_q(n,r) \to \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$, all irreducible objects, Δ-objects and ∇-objects in comod-$A_q(n,r)$ go to the corresponding objects in $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$. So, by the Comparison Theorem [PS1, (5.8)], the exact functor comod-$A(n,r) \to$ comod-$\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$ is an equivalence of categories, and carries indecomposable injective objects to indecomposable injective objects. But $A_q(n,r)$ is a direct sum of indecomposable injective objects, the multiplicity of an indecomposable injective object being the dimension of its socle. The coalgebra $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q]$ has the same decomposition. Thus, they have the same dimension, proving the first isomorphism in (6.11.1).

Similarly, we have an injective homomorphism $B_q(n,r) \hookrightarrow \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]$. But (6.8) gives a surjective homomorphism $\Psi\colon A_q(n,r) \cong \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{U}_q] \twoheadrightarrow \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}]$. However, Ψ is induced by the canonical homomorphism $\mathsf{K}[\mathfrak{U}_q] \to \mathsf{K}[\mathfrak{B}_q]$, sending x_{ij} with $i > j$ to 0. Thus, Ψ factors through $B_q(n,r)$, giving a surjective homomorphism $B_q(n,r) \twoheadrightarrow \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]$. This forces $B_q(n,r) \cong \mathcal{O}_\Gamma\mathsf{K}[\mathfrak{B}_q]$, so (6.11.1) is proved.

As a corollary of (6.11.1), the following result is a special case of (6.10):

The algebra $S_q^+(n,r)$, using the excellent partial ordering on its weight poset, is an excellent Borel subalgebra of the classical q-Schur algebra $S_q(n,r)$.

* The algebra $\widetilde{\mathfrak{U}}_q$ is the algebra $U \otimes_{\mathcal{A}} \mathsf{K}$, where $\mathcal{A}$ and U are defined in [Du, (2.3)]; while the algebra $\mathfrak{U}_q$ is the algebra $'U \otimes_{\mathcal{A}} \mathsf{K}$ in the notation of [Du].

Similarly, $S_q^-(n,r)^{\mathrm{op}}$ is an excellent Borel subalgebra of $S_q(n,r)^{\mathrm{op}}$. Then (2.11) gives a "triangular factorization" $S_q(n,r) = S_q^-(n,r)S_q^+(n,r)$, for $S_q(n,r)$. More precisely,

$$S_q(n,r) = \bigcup_{\lambda\in\Gamma^+} S_q^-(n,r)e_\lambda S_q^+(n,r),$$

where $e_\lambda \in S_q^0(n,r) = S_q^-(n,r)\cap S_q^+(n,r)$ is the primitive idempotent corresponding to λ. This factorization was first obtained in [G] for Schur algebras (the $q=1$ case); it was generalized to q-Schur algebras in [PW, (11.6.1)].

Using (2.11), the factorization extends generalized q-Schur algebras. To do this, return to the setting before (6.11). Consider the "Borel" subalgebra $\mathfrak{B}_q^-$ of $\mathfrak{U}_q$ corresponding to negative roots, and define $\mathfrak{B}_q^-\langle\Gamma\rangle = (\mathcal{O}_\Gamma \mathsf{K}[\mathfrak{B}_q^-])^*$. Similarly, $\mathfrak{B}_q^-\langle\Gamma\rangle^{\mathrm{op}}$ is an excellent Borel subalgebra of $\mathfrak{U}_q\langle\Gamma^+\rangle^{\mathrm{op}}$.

Let $\mathfrak{H}_q\langle\Gamma\rangle = (\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{H}_q])^*$. Then $\mathcal{O}_\Gamma\mathsf{K}[\mathfrak{H}_q]$ has a basis Γ consisting of group-like elements. The dual basis $\{e_\lambda \mid \lambda\in\Gamma\}$ of $\mathfrak{H}_q\langle\Gamma\rangle$ consists of orthogonal primitive idempotents. The natural homomorphisms $\mathsf{K}[\mathfrak{B}_q]\to\mathsf{K}[\mathfrak{H}_q]$ and $\mathsf{K}[\mathfrak{B}_q^-]\to\mathsf{K}[\mathfrak{H}_q]$ are surjective. Hence, $\mathfrak{H}_q\langle\Gamma\rangle \subseteq \mathfrak{B}_q^-\langle\Gamma\rangle\cap\mathfrak{B}_q\langle\Gamma\rangle$. By (2.11), we obtain: result.

(6.12) COROLLARY. *With the above notation, $\mathfrak{U}_q\langle\Gamma^+\rangle = \mathfrak{B}_q^-\langle\Gamma\rangle\mathfrak{B}_q\langle\Gamma\rangle$, or more precisely,*

$$\mathfrak{U}_q\langle\Gamma^+\rangle = \bigcup_{\lambda\in\Gamma^+}\mathfrak{B}_q^-\langle\Gamma\rangle e_\lambda\mathfrak{B}_q\langle\Gamma\rangle.$$

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Added in proof: [K3] has now appeared in *Adv. Math.* **147** (1999), 110–137. Our results do not depend on this paper, whose proofs we have not checked.